Probability (Math 4810/5310)

SAMPLE QUESTIONS for EXAM 2. SOLUTION KEY

1. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

   If $E(X) = 3/5$, find a and b.

   Answer. You need $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 xf(x)\,dx = 1$. Thus, $a = 3/5$, $b = 6/5$.

2. If $E(X) = 2$ and $E(X^2) = 8$, calculate $E((2+4X)^2)$.

   Answer. $E((2+4X)^2) = 4 + 16E(X) + 16E(X^2) = 164$.

3. Suppose X is a random variable with mean and variance both equal to 20. What can be said about $P(0 \le X \le 40)$?

   Answer. $P(0 \le X \le 40) = P(-20 \le X - 20 \le 20) = 1 - P(|X-20| > 20) \ge 1 - 20/20^2 = .95$.

4. Let X be a binomial random variable with $E(X) = 7$ and $V(X) = 2.1$. Find $P(X = 4)$.

   Answer. $np = 7$, $np(1-p) = 2.1$, so $p = .7$ and $n = 10$. $P(X = 4) = \binom{10}{4} .7^4 .3^6$.

5. The probability of error in the transmission of a binary digit over a communication channel is 10^{-3}. Find the probability of more than 3 errors when transmitting 10^3 bits.

   Answer. The distribution of errors is approximately Poisson with parameter $\lambda = 10^{-3} \cdot 10^3 = 1$. Thus, $P(X > 3) = 1 - P(X=0) - P(X=1) - P(X=2) = 1 - 2.5 \cdot e^{-1}$.

6. The lifetime of a color television picture tube is a normal random variable with mean 8.2 years and standard deviation 1.4 years. What percentaghe of such tubes lasts for more than 10 years ?

   Answer. Let X be the lifetime. $P(X > 10) = P((X - 8.2)/1.4 > (10 - 8.2)/1.4) = P(Z > 18/14) \approx .1$.

7. Suppose X has a cdf $F(x) = 1 - \exp(-x^3)$ when $x > 0$ and $F(x) = 0$ when $x \le 0$. Find the probability density function of X^2.

   Answer. $f(x) = 0$ if $x < 0$. If $x > 0$, $f(x) = dF(\sqrt{x})/dx = (3/2)x^{1/2}\exp(-x^{3/2})$.

8. Let X and Y be independent random variables each of which is geometrically distributed with parameter p. Find $P(X + Y = k)$.

   Answer. The sum is negative binomial and $P(X + Y = k) = u(k, 2, p) = (k-1)p(1-p)^{k-1}$.

9. Suppose that X is an exponentially distributed random variable with parameter 1, that Y is a random variable uniformly distributed on $[-3,-1]$, and that X and Y are independent. Find the expectation of $|X+Y|$.

Answer. The cdf of $Z = |X+Y|$ is, for $z \geq 0$, $F(z) = \int_{|x+y|\leq z,\, x\geq 0,\, -3\leq y\leq -1} e^{-x}\frac{1}{2}\,dx\,dy$. The pdf is

$$f(z) = \begin{cases} (e^{-1}-e^{-3})e^{z}, & 0 \leq z \leq 1, \\ 1/2 + (1/2)e^{-z}e^{-1} - e^{-z}e^{-3}, & 1 \leq z \leq 3, \\ (e^{-1}-e^{-3})e^{-z}/2, & z \geq 3. \end{cases}$$

Thus,

$$E(Z) = \int_0^\infty z f(z)\,dz = (e^{-1}-e^{-3})(1+e^{-1}) + 1.$$

10. Given that X and Y are independent exponentially distributed random variables with parameters λ and μ respectively, evaluate $V(XY)$.

Answer. $V(XY) = E(X^2Y^2) - (E(XY))^2 = E(X^2)E(Y^2) - (E(X)E(Y))^2 = (2/\lambda^2)(2/\mu^2) - (1/\lambda \cdot 1/\mu)^2 = 3/(\lambda^2\mu^2)$.