Mathematical Probability (Math 6383)

FINAL EXAM. May 7, 2008.

Instructions: this is a take-home test. It is to be turned in on May 14. Each problem is worth 20 pt, parts of a problem having equal weights. Please, write legibly and substantiate your argument.

1. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Prove that the set $\{\omega \in \Omega : X(\omega) = Y(\omega)\}$ is an element of $\mathcal{F}$.

2. Suppose X and Y are independent random variables. Show that if X and $X+Y$ are identically distributed, then $Y = 0$ a.s. (Hint: look at characteristic functions.)

3. Suppose $X_1, X_2, \ldots$ are i.i.d. lognormal random variables. Prove that the $(\prod_{i=1}^n X_i)^{1/n}$ converge a.s. and identify the limit in terms of the distribution of the X_i. (Hint. Take log)

4. Let X_i, , $i = 1, 2, \ldots$, be independent with $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 = 1$. Introduce $Y_m = \sum_{n=1}^m X_n/n$. Show the Y_m converge in probability. (Hint. Show that it is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Use completeness arguments to deduce convergence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$.)

5. Let X_i , $i = 1, 2, \ldots$, be independent $N(\mu, \sigma^2)$-random variables and $\mathcal{F}_n$ denote the σ-algebra generated by $X_1, \ldots, X_n$. Let $\lambda \in \mathbb{R}$. Prove that $(\exp(\lambda \sum_{i=1}^n X_i - \lambda n\mu - (\lambda^2/2)n\sigma^2))_{n\geq 0}$ is a martingale relative to $(\mathcal{F}_n)_{n\geq 0}$.