Mathematical Probability (Math 6383)

FINAL EXAM. SOLUTIONS

1. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Prove that the set $\{\omega \in \Omega : X(\omega) = Y(\omega)\}$ is an element of $\mathcal{F}$.

   Solution. X and Y are random variables, so $X - Y$ is a random variable. The set $\{\omega : X(\omega) - Y(\omega) = 0\}$ is the inverse image under $X - Y$ of the set $\{0\}$ which is a Borel set. Hence, $\{\omega : X(\omega) - Y(\omega) = 0\}$ is an element of $\mathcal{F}$.

2. Suppose X and Y are independent random variables. Show that if X and $X + Y$ are identically distributed, then $Y = 0$ a.s. (Hint: look at characteristic functions.)

   Solution. Since X and Y are independent, for $u \in \mathbb{R}$,

   $$\mathbf{E}e^{iu(X+Y)} = \mathbf{E}e^{iuX}e^{iuY} = \mathbf{E}e^{iuX}\mathbf{E}e^{iuY}\,.$$

   On the other hand, $\mathbf{E}e^{iu(X+Y)} = \mathbf{E}e^{iuX}$ because X and $X + Y$ are identically distributed. Hence, $\mathbf{E}e^{iuY} = 1$. Since 1 is the characteristic function of the constant 0 and there is one-to-one corresponedence between random variables and their characteristic functions, $Y = 0$ a.s.

3. Suppose $X_1, X_2, \ldots$ are i.i.d. lognormal random variables. Prove that the $(\prod_{i=1}^n X_i)^{1/n}$ converge a.s. and identify the limit in terms of the distribution of the X_i. (Hint. Take log)

   Solution. Since the X_i are lognormal, the random variables $\log X_i$ are normally distributed, say, with mean μ and variance σ^2. By the strong law of large numbers, as $n \to \infty$ a.s.

   $$\frac{1}{n}\sum_{i=1}^n \log X_i \to \mu\,.$$

   Since e^x is a continuous function,

   $$(\prod_{i=1}^n X_i)^{1/n} = \exp(\frac{1}{n}\sum_{i=1}^n \log X_i) \to e^{\mu}\,.$$

4. Let X_i, $, i = 1, 2, \ldots,$ be independent with $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 = 1$. Introduce $Y_m = \sum_{n=1}^m X_n/n$. Show the Y_m converge in probability. (Hint. Show that it is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Use completeness arguments to deduce convergence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$.)

   Solution. We have, for $m > k$, by the fact that the X_n are independent with zero mean

   $$\mathbf{E}(Y_m - Y_k)^2 = \mathbf{E}(\sum_{n=k+1}^m X_n/n)^2 = \sum_{n=k+1}^m \mathbf{E}X_n^2/n^2 = \sum_{n=k+1}^m 1/n^2\,.$$

   The latter sum tends to zero as $m \to \infty$ and $k \to \infty$. It follows that Y_m is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. By completeness of $L^2(\Omega, \mathcal{F}, \mathbf{P})$, the Y_m

converge to a random variable Y. Convergence in L^2 imples convergence in probability, so the Y_m converge to Y in probability.

An alternative solution. Since the X_n/n are independent with zero mean and $\sum_{n=1}^{\infty} \mathbf{E}(X_n/n)^2 = \sum_{n=1}^{\infty} 1/n^2 < \infty$, by Theorem 27.4 the Y_m converge a.s. Since a.s. convergence is stronger than convergence in probability, they converge in probability too.

5. Let X_i , $i = 1, 2, \ldots$, be independent $N(\mu, \sigma^2)$-random variables and $\mathcal{F}_n$ denote the σ-algebra generated by $X_1, \ldots, X_n$. Let $\lambda \in \mathbb{R}$. Prove that $(\exp(\lambda \sum_{i=1}^{n} X_i - \lambda n\mu - (\lambda^2/2)n\sigma^2))_{n\geq 0}$ is a martingale relative to $(\mathcal{F}_n)_{n\geq 0}$.

   Solution. Let

$$M_n = \exp(\lambda \sum_{i=1}^{n} X_i - \lambda n\mu - (\lambda^2/2)n\sigma^2)\,.$$

   Since X_i is $\mathcal{F}_n$-measurable for $i \leq n$, $\sum_{i=1}^{n} X_i$ is $\mathcal{F}_n$-measurable. Therefore, M_n, being a measurable function of $\sum_{i=1}^{n} X_i$, is $\mathcal{F}_n$-measurable.

   Next, since the X_i are i.i.d.

$$\mathbf{E}M_n = \mathbf{E}\exp(\lambda \sum_{i=1}^{n} X_i - \lambda n\mu - (\lambda^2/2)n\sigma^2) = (\mathbf{E}\exp(\lambda X_1))^n \exp(-\lambda n\mu - (\lambda^2/2)n\sigma^2) < \infty$$

   because a normally distributed random variable has finite exponential moments of any order.

   It remains to check the martingale property. We have

$$\mathbf{E}(M_{n+1}|\mathcal{F}_n) = \mathbf{E}(\exp(\lambda \sum_{i=1}^{n+1} X_i - \lambda(n+1)\mu - (\lambda^2/2)(n+1)\sigma^2)|\mathcal{F}_n)$$
$$= \mathbf{E}(\exp(\lambda \sum_{i=1}^{n} X_i - \lambda n\mu - (\lambda^2/2)n\sigma^2)\exp(\lambda X_{n+1} - \lambda\mu - (\lambda^2/2)\sigma^2)|\mathcal{F}_n)$$
$$= \mathbf{E}(M_n \exp(\lambda X_{n+1} - \lambda\mu - (\lambda^2/2)\sigma^2)|\mathcal{F}_n)\,.$$

   Since M_n is $\mathcal{F}_n$–measurable and X_{n+1} is independent of $\mathcal{F}_n$, by the properties of conditional expectations

$$\mathbf{E}(M_n \exp(\lambda X_{n+1} - \lambda\mu - (\lambda^2/2)\sigma^2)|\mathcal{F}_n) = M_n \mathbf{E}(\exp(\lambda X_{n+1} - \lambda\mu - (\lambda^2/2)\sigma^2)|\mathcal{F}_n)$$
$$= M_n \mathbf{E}\exp(\lambda X_{n+1} - \lambda\mu - (\lambda^2/2)\sigma^2)\,.$$

   Since X_{n+1} is normal with mean μ and variance σ^2, $\mathbf{E}\exp(\lambda X_{n+1}) = \exp(\lambda\mu + (\lambda^2/2)\sigma^2)$ so that $\mathbf{E}(M_{n+1}|\mathcal{F}_n) = M_n$ as needed.