

## Solutions to Assignment #04 – MATH 3250

Kawai

(#1) **Tractrix.** The boat starts at  $(0, 20)$  and Zorba walks along the positive x-axis.

In my version, Zorba takes tiny 0.1 unit steps on the x-axis dock.

Every time I calculate a new  $(x, y)$  position for the boat, I guarantee that the new position will be exactly 20 units from Zorba's position.

Here are your tasks:

- (a) Run my code. It will tell you that based on the value of the last y-coordinate position that I calculated, there was an associated theoretical value of  $x$  (on p. 68). I calculated the error in  $x$ , assuming that my y-coordinate was correct (which it really is not, but it is close).

The error `deltax` is equal to the  $(\textit{calculated last } x) - (\textit{theoretical last } x)$ .

Note that the last calculated  $x$  is the 1001st element in the array `xlist`.

Report the value of `deltax`.

The estimated error is 0.214601. We can do better than that by taking smaller steps.

- (b) Now modify my code so that Zorba takes tinier steps of length 0.01.

- (i) We changed Line #17:

```
for z = 0.01:0.01:100
```

This will give us 10000 steps and the last calculated  $x$  should be in the 10001st element in the array `xlist`.

- (ii) Lines #37, #40, and #41 must fetch that last element.

The estimated error is 0.021527.

This is actually bad news. We did 10 times the amount of work and only improved our accuracy by one decimal point.

Suppose we process 100000 steps. It turns out, then, the estimated error is 0.002153. Again, for 10 times the amount of work, we pick up one decimal point of accuracy.

- (c) Explain how I am calculating the new positions  $(x_{new}, y_{new})$ .

Clearly, the value of  $z$  in the program is Zorba's position at  $(z, 0)$ .

Since the current position of the boat is  $(x, y)$ , the vector from Zorba ( $Z$ ) to the boat ( $B$ ) must be

$$\overrightarrow{ZB} = \langle x - z, y \rangle.$$

Thus, the distance (norm) between them is

$$\text{dist} = \|\overrightarrow{ZB}\| = \sqrt{(x - z)^2 + y^2}$$

and the associated unit vector is

$$\langle \text{unitvector}_x, \text{unitvector}_y \rangle = \left\langle \frac{x - z}{\text{dist}}, \frac{y}{\text{dist}} \right\rangle.$$

The rope is exactly 20 units long, so our new position will be 20 units from Zorba  $(z, 0)$ , in the direction of the unit vector.

Thus, the new coordinate of the boat will be

$$\left( z + 20 * \frac{x - z}{\text{dist}}, 20 * \frac{y}{\text{dist}} \right).$$

In the program, those components were:

`newx = z + 20.* unitvectorx`

`newy = 20.* unitvectory`

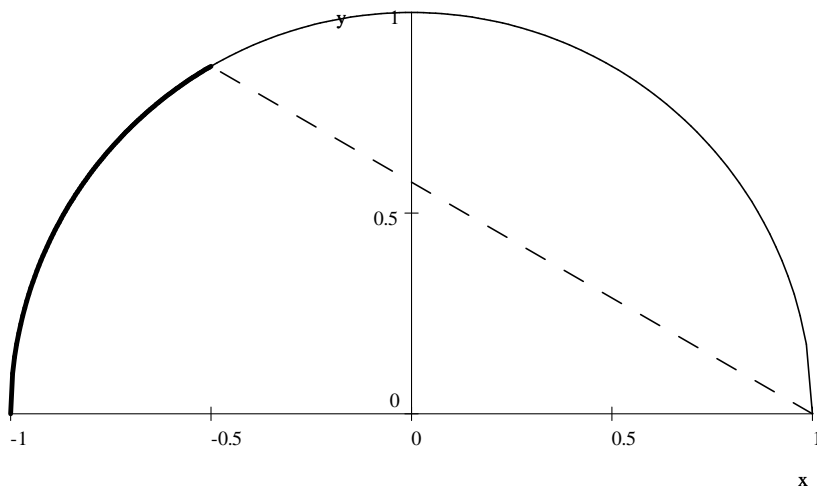
Note during each time segment, the boat moves a different amount (slightly).

If we wanted to estimate the total arc length, we would need to accumulate the line segment distances.

(#2) Solve Problem (7.18).

We decided best set-up is to let the lake be the unit disk  $x^2 + y^2 \leq 1$  (miles).

The swimmer/walker starts at  $(1, 0)$  and then chooses a point along the upper semicircle to swim toward.



In terms of what would be polar coordinates, all of the points on the semicircle are of the form

$$(r, \theta) = (1, \theta), \text{ where } 0 \leq \theta \leq \pi.$$

Thus, whenever we choose an angle, we determine a swimming/walking path.

Using the Law of Cosines, the swimming distance is

$$d_s = \sqrt{1^2 + 1^2 - 2(1)(1)\cos(\theta)} = \sqrt{2 - 2\cos(\theta)}.$$

The remaining arc is the walking distance ( $s = r\theta$ ):

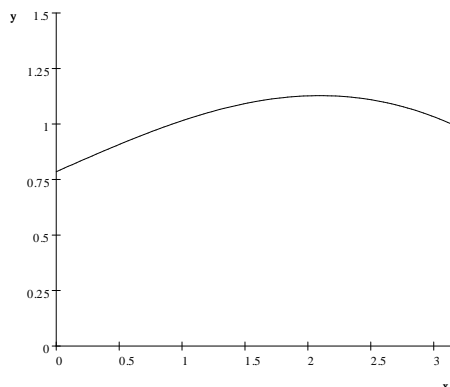
$$d_w = (1)(\pi - \theta) = \pi - \theta.$$

The total of amount of time expended is

$$f(\theta) = \frac{\sqrt{2 - 2\cos(\theta)}}{v_s} + \frac{\pi - \theta}{v_w}.$$

(a) If  $v_s = 2$  mph and  $v_w = 4$  mph, then the total time is

$$f(\theta) = \frac{\sqrt{2 - 2\cos(\theta)}}{2} + \frac{\pi - \theta}{4}$$



The function is concave *downward*, so we can find the maximum amount of time by taking the derivative and setting it equal to zero.

$$\begin{aligned} f'(\theta) &= \frac{1}{2} \left( \frac{1}{2} (2 - 2\cos(\theta))^{-1/2} (2\sin(\theta)) \right) - \frac{1}{4} \\ &= \frac{\sin(\theta)}{2\sqrt{2 - 2\cos(\theta)}} - \frac{1}{4} = 0. \end{aligned}$$

Thus, we can numerically solve

$$\begin{aligned} \frac{\sin(\theta)}{2\sqrt{2 - 2\cos(\theta)}} &= \frac{1}{4} \Rightarrow 4\sin(\theta) = 2\sqrt{2 - 2\cos(\theta)} \\ 16\sin^2(\theta) &= 4(2 - 2\cos(\theta)) \Rightarrow 2\sin^2(\theta) = 1 - \cos(\theta) \end{aligned}$$

$$2(1 - \cos^2(\theta)) = 1 - \cos(\theta) \Rightarrow 2 - 2\cos^2(\theta) = 1 - \cos(\theta)$$

$$2\cos^2(\theta) - \cos(\theta) - 1 = 0 \Rightarrow (2\cos(\theta) + 1)(\cos(\theta) - 1) = 0$$

$$\cos(\theta) = -\frac{1}{2}, 1 \Rightarrow \theta = \frac{2\pi}{3}, 0.$$

The solution  $\theta = 0$  is not allowed since  $\sqrt{2 - 2\cos(\theta)}$  appears in the denominator of one of the terms.

Thus,  $\theta = \frac{2\pi}{3}$  gives us the maximum time. We can check this by finding  $f''(\theta)$ . [Too long. I used Derive5.]

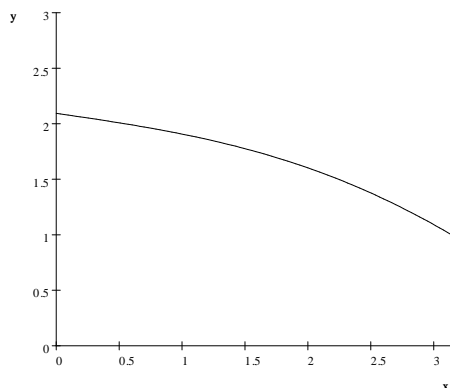
The maximum time is  $f\left(\frac{2\pi}{3}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} \doteq 1.13$  hours.

The minimum time is located at one of the interval end points. From the first function graph, we see that the minimum time occurs when  $\theta = 0$ . In other words, he should not swim at all, and walk the entire semicircle.

$$f(0) = \frac{\pi}{4} \text{ hours.}$$

- (b) If  $v_s = 2$  mph and  $v_w = \frac{3}{2}$  mph, then we have

$$f(\theta) = \frac{\sqrt{2 - 2 \cos(\theta)}}{2} + \frac{\pi - \theta}{(3/2)} = \frac{\sqrt{2 - 2 \cos(\theta)}}{2} + \left(\frac{2}{3}\right)(\pi - \theta)$$



There is no local maximum and the curve is concave downward on  $[0, \pi]$ .

The maximum time occurs when  $\theta = 0$  and the minimum time occurs when  $\theta = \pi$ .

$$f(0) = \frac{2}{3}\pi \doteq 2.094 \text{ hours.}$$

$$f(\pi) = 1 \text{ hour.}$$

- (c) If he swims at 2 mph, what is the minimum walking speed for which it is quickest to walk the entire distance. In other words, for what values of  $v_w$  does  $\theta = 0$  give us the minimum total time?

We see that it should be less than 4 mph, since from part (a), he walks the entire semicircle when  $v_w = 4$  mph.

Since the minimum time always occurs at one of the end points, we see that if the total time at both end points were the same, then we would have the answer. When  $v_s = 2$ , then we have

$$f(\theta) = \frac{\sqrt{2 - 2 \cos(\theta)}}{2} + \frac{\pi - \theta}{v_w}$$

$$[\text{swimming only}] \quad f(0) = \frac{\pi}{v_w} \text{ hours}$$

$$[\text{walking only}] \quad f(\pi) = 1 \text{ hour.}$$

These times are equal when  $v_w = \pi$  mph.

If  $v_w = \pi$  mph, then either path gives the minimum time. If  $v_w > \pi$  mph, then he should walk the entire distance.

- (d) Is there any combination of swimming and walking speeds for which there is a *mixed* strategy?

NO. We can show that the total time function is ALWAYS concave downward, and thus, we can never find a  $\theta$  on the interior of  $[0, \pi]$  which gives us a local minimum.

Let  $k = \frac{v_s}{v_w} > 0$ . Since  $v_s$  is a constant, let  $g(\theta) = v_s * f(\theta)$ .

Since they only differ by a positive multiplicative constant, the maximums will occur at the same value of  $\theta$ .

$$\begin{aligned} f(\theta) &= \frac{\sqrt{2 - 2 \cos(\theta)}}{v_s} + \frac{\pi - \theta}{v_w} \\ v_s * f(\theta) &= \sqrt{2 - 2 \cos(\theta)} + \left(\frac{v_s}{v_w}\right)(\pi - \theta) \\ &= \sqrt{2}\sqrt{1 - \cos(\theta)} + k(\pi - \theta) \end{aligned}$$

If we take the second derivative, we have

$$\begin{aligned}
[g(\theta)]'' &= [g'(\theta)]' = \frac{d}{d\theta} \left[ \frac{\sqrt{2}}{2} \left( \frac{\sin(\theta)}{\sqrt{1-\cos(\theta)}} \right) \right] \\
&= \frac{\sqrt{2}}{2} * \frac{d}{d\theta} \left[ \frac{\sin(\theta)}{\sqrt{1-\cos(\theta)}} \right] \\
&= \frac{\sqrt{2}}{2} \left( \frac{\sqrt{1-\cos(\theta)}(\cos(\theta)) - \sin(\theta) \left( \frac{1}{2} \left( \frac{\sin(\theta)}{\sqrt{1-\cos(\theta)}} \right) \right)}{\left(\sqrt{1-\cos(\theta)}\right)^2} \right) \\
&= \frac{\sqrt{2}}{2} \left( \frac{\frac{2(1-\cos(\theta))(\cos(\theta))}{2\sqrt{1-\cos(\theta)}} - \frac{\sin^2(\theta)}{2\sqrt{1-\cos(\theta)}}}{1-\cos(\theta)} \right) \\
&= \frac{\sqrt{2}}{2} \left( \frac{2\cos(\theta) - 2\cos^2(\theta) - \sin^2(\theta)}{2(1-\cos(\theta))^{3/2}} \right) = \frac{\sqrt{2}}{4} \left( \frac{2\cos(\theta) - 2\cos^2(\theta) - (1-\cos^2(\theta))}{(1-\cos(\theta))^{3/2}} \right) \\
&= -\frac{\sqrt{2}}{4} \left( \frac{\cos^2(\theta) - 2\cos(\theta) + 1}{(1-\cos(\theta))^{3/2}} \right) = -\frac{\sqrt{2}}{4} \left( \frac{(\cos(\theta) - 1)^2}{(1-\cos(\theta))^{3/2}} \right)
\end{aligned}$$

We see that  $(\cos(\theta) - 1)^2$  is a perfect square and  $(1 - \cos(\theta))^{3/2}$  is the square root of a cube. Both of these must be nonnegative.

These are multiplied by a negative multiplicative factor. Thus, the second derivative is always negative and regardless of the ratio of the swimming/walking speeds! Thus, we can never find a mixed strategy which minimizes the total time.