Notesheet for Calculus III Final – MATH 2421

Summer 2005

1. 2-D and 3-D Vectors

   If $\mathbf{u} = \langle a, b \rangle$, then $\tan(\theta_{\mathbf{u}}) = \frac{b}{a}$. The direction angle is $\theta_{\mathbf{u}}$.

   The unit direction vector associated with nonzero $\mathbf{u}$ is $\frac{\mathbf{u}}{|\mathbf{u}|}$.

   $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| * |\mathbf{v}| * \cos(\alpha)$. The angle of separation is α.

   $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$. $\mathbf{u} \cdot \mathbf{v} = 0$ when $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| * |\mathbf{v}| * \sin(\alpha)$. The angle of separation is α.

   $\mathbf{Proj_v u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$

   If a plane in space has normal vector $\langle a, b, c \rangle$ and it passes through the point (x_0, y_0, z_0), then its standard form is

   $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ellipsoid. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Elliptic paraboloid.

   $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$. Hyperbolic paraboloid. (Saddle-shape.) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$. Elliptic cones.

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Hyperboloid of ONE sheet. (One negative sign.)

2. Conversions:

   $x = r\cos(\theta)$ $y = r\sin(\theta)$ $\tan(\theta) = \frac{y}{x}$

   $r = \sqrt{x^2 + y^2}$ $r^2 = x^2 + y^2$

   $\rho = \sqrt{x^2 + y^2 + z^2}$ $\rho^2 = x^2 + y^2 + z^2$

   $z = \rho\cos(\phi)$ $r = \rho\sin(\phi)$ $r^2 = \rho^2\sin^2(\phi)$ $\cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$.

   $r = a\ constant$ is a circular cylinder.

   $\rho = a\ constant$ is a sphere when $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

   $\phi = a\ constant\ angle$ is a cone when $\rho \ge 0$ and $0 \le \theta \le 2\pi$.

3. Arc length, unit normal, and unit tangent.

   If the position function is $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$.

   $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

4. Multivariable Functions and Partial Derivatives

   Total differential: $dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy = z_x dx + z_y dy$

   Related rates: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ (You can extend this to three variables...)

   $\nabla f(x, y, z) = \langle f_x,\ f_y,\ f_z \rangle$. The gradient vector always points in the direction of the greatest directional derivative (best rate of increase in f).

   If $\mathbf{u}$ is a unit direction vector $\langle \cos(\theta), \sin(\theta) \rangle$, then the directional derivative is

   $D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta) = \nabla f(x, y) \cdot \mathbf{u}$.

   If we have a surface $F(x, y, z) = 0$, then the implicit derivative $\frac{\partial z}{\partial x} = -\left(\frac{F_x}{F_z}\right)$.

   If $\nabla f = \mathbf{0}$ or any of the components of ∇f is undefined, then we have a critical point.

   For 2-D functions, we have $d = f_{xx} f_{yy} - (f_{xy})^2$.

   If $d > 0$ and $f_{xx} > 0$, we have a relative minimum.

   If $d > 0$ and $f_{xx} < 0$, we have a relative maximum.

   If $d < 0$, we have a saddle point.

5. "Del" operator $= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ is a vector.

   $\nabla f(x, y, z) = \langle f_x,\ f_y,\ f_z \rangle$ is a vector field.

   $\nabla \cdot \langle P, Q, R \rangle = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$.

   $\nabla \times \langle P, Q, R \rangle = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$.

6. Jacobians:

   $dA = dy\ dx = r\ dr\ d\theta$

   $dV = dz\ dy\ dx = r\ dz\ dr\ d\theta = \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

7. Conservative Field

   A 2-D vector field $\mathbf{F}(x, y) = \langle P, Q \rangle$ is conservative if and only if $Q_x - P_y = 0$.

   A 3-D vector field $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ is conservative if and only if $\nabla \times \mathbf{F} = \langle 0, 0, 0 \rangle$.

   If a vector field $\mathbf{F}$ is conservative, then the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the starting point and the ending point of the curve C (path independent).

   Furthermore, there must be a potential (scalar) function f such that $\nabla f = \mathbf{F}$ and the value of the line integral must be

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\textit{ending point}) - f(\textit{starting point}).$$

8. Green's Theorem

   If C is a simple closed curve and R is the interior of C, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (Q_x - P_y)\ dA,$$

   where the closed line integral is traced once around C counterclockwise. **F** must be continuous in the interior of C.

9. Scalar Surface Integral

   If S is some surface defined by $z = g(x, y)$, then let D be the projection of S down onto the xy-plane.

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2}\, dA$$

10. Flux Surface Integral

    Consider the vector field $\mathbf{F} = \langle P, Q, R \rangle$. If S is some surface defined by $z = g(x, y)$, then let D be the projection of S down onto the xy-plane. If S is "oriented-upward", e.g., you are trying to find the net flux passing upward through S, then you want

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\ dS = \iint_D (-P g_x - Q g_y + R)\, dA.$$

11. Divergence Theorem

    If S is a simple closed surface and Q is its interior, then the net outward flux through S is

$$\oint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_Q (\nabla \cdot \mathbf{F})\ dV.$$

    It is assumed that **n** is the outward unit normal vector. **F** must be continuous in the interior of S.