Solutions to Second Test for Calculus III – MATH 2421
Form A – Summer 2005

Kawai

(#1) Suppose we have a vector field $\mathbf{G}(x,y,z) = \langle P, Q, R\rangle$ and all the first and second partials of P, Q, and R are continuous (Clairaut holds).

Which field below is equivalent to $\nabla \cdot (\nabla \times \mathbf{G})$?

The answer must be **(f)** zero. It's the only scalar quantity on the list. Remember that this was a divergence of a curl.

$\nabla \times \mathbf{G} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{G}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle \\ &= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}) = 0.\checkmark\end{aligned}$$

(#2) If C is the line segment which starts at $(2,3,-5)$ and ends at $(2,3,4)$, then which of these quantities MUST equal $\int_C P dx + Q dy + R dz$?

We see that x and y do not change, so $dx = dy = 0$. The only remaining term is $\int_C R dz$.

The correct answer is **(d)**.

(#3) $\nabla \times \langle xz,\ xy,\ yz\rangle = \left\langle \frac{\partial}{\partial y}[yz] - \frac{\partial}{\partial z}[xy],\ \frac{\partial}{\partial z}[xz] - \frac{\partial}{\partial x}[yz],\ \frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[xz]\right\rangle =$

$\langle z,\ x,\ y\rangle$. The correct answer is **(c)**.

(#4) True or False?

(a) Green's Theorem will work when the field is $\mathbf{F} = \left\langle \frac{1}{y}, \frac{1}{x}\right\rangle$ and C is $x^2 + y^2 = 1$.

FALSE. The vector field is undefined at $(0,0)$, which is inside the boundary curve.

(b)

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \left(x^2 + y^2\right)^{3/2} dy\ dx = \int_0^{2\pi} \int_0^5 r^4\ dr\ d\theta.$$

FALSE. The integral on the left only uses the portion of the circle $x^2 + y^2 \leq 25$ in Quadrant I.

Thus, we have

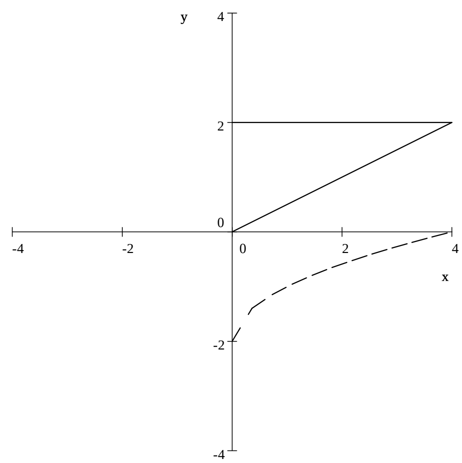
$r: \quad 0 \to 5$

$\theta: \quad 0 \to \pi/2.$

(#5) Suppose we have the region D: $\frac{x}{2} \le y \le 2$, for $0 \le x \le 4$.

Our objective function is $f(x, y) = y - \sqrt{x}$. The absolute minimum point must be somewhere along the boundary.

The level curves of f are square root function graphs. We want to find the *first* level curve which intersects D.



(a) Sketch in the level curve which corresponds to $k = -2$. We sketch $y = \sqrt{x} - 2$.

(b) The Extreme Sports Fairy visits you and tells you that the absolute minimum point is located along the diagonal edge.
Find that point, AND find the lowest value of f over the region D.

The equation for the diagonal edge is $y = \frac{x}{2}$. We parameterize this edge.

$$\mathbf{r}(t) = \left\langle t, \frac{t}{2} \right\rangle, \ 0 \le t \le 4.$$

We substitute this into our objective function.

$$f\left(t, \frac{t}{2}\right) = \frac{t}{2} - \sqrt{t} \Rightarrow f' = \frac{1}{2} - \frac{1}{2\sqrt{t}} = 0 \Rightarrow t = 1.$$

We see that f'' is positive when $t = 1$, so this must be a relative minimum along that edge.

The minimum point is $\left(1, \frac{1}{2}\right)$, and the absolute minimum value of f is $\frac{1}{2} - \sqrt{1} = -\frac{1}{2}$.

Thus, the level curve $y - \sqrt{x} = -\frac{1}{2}$ is the first level curve to intersect our domain of interest.

(#6) Due to the current order of integration, the following double integral is impossible to evaluate:

$$\int_0^1 \int_0^{\cos^{-1}(y)} \sqrt{\sin(x)} \, dx \, dy.$$

Use Fubini's Theorem to evaluate this double integral. I just *love* u-substitutions, don't you?

We need to go vertically first.

$x = \cos^{-1}(y) \Rightarrow y = \cos(x)$.

Thus, the lower function is $y = 0$ and the upper function is $y = \cos(x)$.

$y: \quad 0 \to \cos(x)$

$x: \quad 0 \to \pi/2$.

$$\int_0^{\pi/2} \int_0^{\cos(x)} \sqrt{\sin(x)}\ dy\ dx = \frac{2}{3}.$$

Inner: $\sqrt{\sin(x)} \int_0^{\cos(x)} dy = \cos(x) \sqrt{\sin(x)}$.

Outer: $\int_0^{\pi/2} \cos(x) \sqrt{\sin(x)}\ dx$. Let $u = \sin(x)$, $du = \cos(x)\, dx$.

$\int u^{1/2} du \Rightarrow \frac{2}{3} u^{3/2} \Rightarrow \frac{2}{3} \left[(\sin(x))^{3/2} \right]_0^{\pi/2} = \frac{2}{3} \left(1^{3/2} - 0^{3/2} \right) = \frac{2}{3}. \checkmark$

(#7) Suppose $\mathbf{F}(x, y) = \left\langle 2x + \ln(y),\ \frac{1}{y^2} + \frac{x}{y} \right\rangle$.

(a) Use the appropriate test and show that $\mathbf{F}$ is conservative. Find a potential function $f(x, y)$.

Show that $Q_x = P_y$.

$Q_x = \frac{\partial}{\partial x} \left[\frac{1}{y^2} + \frac{x}{y} \right] = \frac{1}{y}$.

$P_y = \frac{\partial}{\partial y} [2x + \ln(y)] = \frac{1}{y}. \checkmark$

We integrate twice to find $f(x, y)$.

$$f(x, y) = \int (2x + \ln(y))\, dx = x^2 + x \ln(y) + \alpha(y).$$

$$f(x, y) = \int \left(\frac{1}{y^2} + \frac{x}{y} \right) dy = -\frac{1}{y} + x \ln(y) + \beta(x).$$

Thus, we see that $\alpha(y)$ must equal $\left(-\frac{1}{y} \right)$ and $\beta(x)$ must equal x^2.

$$f(x, y) = x \ln(y) + x^2 - \frac{1}{y} + C.$$

(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along any simple path which starts at $(1,1)$ and ends at $(3,e)$ and stays in Quadrant I, away from the coordinate axes.

Since the field is conservative, the value of the work integral must be path independent (as long as the path is simple, and it does not cross into unfriendly territory). We have the Fundamental Theorem for Line Integration.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r} = f\,(end) - f\,(start) \\ &= f\,(3,e) - f\,(1,1) \\ &= 3\ln(e) + 3^2 - \frac{1}{e} - \left((1)\ln(1) + 1^2 - \frac{1}{1}\right) = 12 - \frac{1}{e}.\end{aligned}$$

(#8) Find the mass of half of a cantaloupe. The solid region E lies between the spheres $x^2+y^2+z^2\le 1$ and $x^2+y^2+z^2\le 9$, and we have $z\le 0$.

This gives us limits in Spherical Coordinates:

$\rho:\ 1\to 3$

$\phi:\ \pi/2\to\pi$ [This half is below the xy-plane!]

$\theta:\ 0\to 2\pi.$

The density function is $\sigma(x,y,z) = \dfrac{-z}{\sqrt{x^2+y^2+z^2}} \Rightarrow (-\cos(\phi))$.

$$m = \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^3 (-\cos(\phi))\ \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta = \frac{26}{3}\pi.$$

Inner: $\displaystyle\int_1^3 \rho^2\ d\rho = \left[\frac{\rho^3}{3}\right]_1^3 = \frac{26}{3}.$

Next: $\displaystyle -\int_{\pi/2}^{\pi} \sin(\phi)\cos(\phi)\,d\phi \Rightarrow$ Let $u=\sin(\phi)$, $du=\cos(\phi)\,d\phi$.

$$-\int u\ du \Rightarrow -\frac{u^2}{2} \Rightarrow -\frac{1}{2}\left[\sin^2(\phi)\right]_{\pi/2}^{\pi} = \frac{1}{2}.$$

The Outer integral multiplies the previous results by 2π.

The final answer is $\left(\dfrac{26}{3}\right)\left(\dfrac{1}{2}\right)(2\pi) = \dfrac{26}{3}\pi.$✓

(#9) Mass integral. We have a piece of wire:

$y = e^x$, $0 \le x \le 1$. The density function is $\sigma(x, y) = y^2$. Thus, the "heavy" end is at the top of the wire. Find its mass.

The parameterization is $\mathbf{r}(t) = \langle t, e^t \rangle$, $0 \le t \le 1$.

$$ds = \sqrt{(1)^2 + (e^t)^2}dt = \sqrt{1 + e^{2t}}dt.$$

We substitute into the density function and form the mass integral.

$$\int_0^1 (e^t)^2 \sqrt{1 + e^{2t}}dt =$$

Let $u = 1 + e^{2t}$, $du = 2e^{2t}dt \Rightarrow e^{2t}dt = \frac{1}{2}du.$

$$\frac{1}{2}\int u^{1/2}du \Rightarrow \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) \Rightarrow \frac{1}{3}\left[\left(1 + e^{2t}\right)^{3/2}\right]_0^1 = \frac{1}{3}\left(\left(1 + e^2\right)^{3/2} - 2^{3/2}\right) = \frac{1}{3}\left(\left(1 + e^2\right)^{3/2} - 2\sqrt{2}\right).$$

(#10) Matching the scaled characterizations of vector fields. Try evaluating the fields at some obvious points like $(1, 0)$, $(0, 1)$, etc.

(a) Which one of the vector fields most closely resembles $\mathbf{F}(x, y) = \langle xy,\ x \rangle$?
In Quadrant I, the vectors should have a positive horizontal and vertical component.
The correct choice is **(ii)**.

(b) Which one of the vector fields most closely resembles $\mathbf{F}(x, y) = \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}} \right\rangle$?

This is radially INWARD. The correct choice is **(i)**.

(c) Which one of the vector fields most closely resembles $\mathbf{F}(x, y) = \langle x + y,\ x - y \rangle$?
Along the line $y = x$, the vectors should have a zero vertical component (horizontal only). Along the line $y = -x$, the vectors should have a zero horizontal component (vertical only).
The correct choice is **(iv)**.

(#11) Find the work performed by the force field $\mathbf{F}(x, y) = \langle -y,\ y \rangle$.

The path is defined by C: $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$, COUNTERCLOCKWISE.

The parameterization is $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $0 \le t \le \pi/2$. [Quarter circle!]

$d\mathbf{r} = \langle -2\sin(t), 2\cos(t) \rangle\, dt.$

$$\int_0^{\pi/2} \langle -2\sin(t), 2\sin(t) \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle\, dt = \int_0^{\pi/2} \left(4\sin^2(t) + 4\sin(t)\cos(t)\right) dt$$

$$4\int_0^{\pi/2} \left(\sin^2(t) + \sin(t)\cos(t)\right) dt = \pi + 2.$$

$$\int_0^{\pi/2} \sin^2(t)\, dt = \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos(2t)}{2}\right) dt = \left[\frac{t}{2} - \frac{\sin(2t)}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.\checkmark$$

$$\int_0^{\pi/2} \sin(t)\cos(t)\, dt = \left[\frac{\sin^2(t)}{2}\right]_0^{\pi/2} = \frac{1}{2}.\checkmark$$

(#12) Green's Theorem. $\mathbf{F}(x,y) = \langle y^2, x \rangle$

C is the triangle (once around counterclockwise). The interior is bounded by the lines $y = 0$, $x = 1$, and $y = x$.

By Green's Theorem, we have

$$\text{Circulation } = \oint_C P dx + Q dy = \iint_R (Q_x - P_y)\, dA.$$

The interior region R is defined by:

$y: \quad 0 \to x$

$x: \quad 0 \to 1.$

$$Q_x - P_y = 1 - 2y$$

$$\int_0^1 \int_0^x (1-2y)\ dy\ dx = \frac{1}{6}.$$

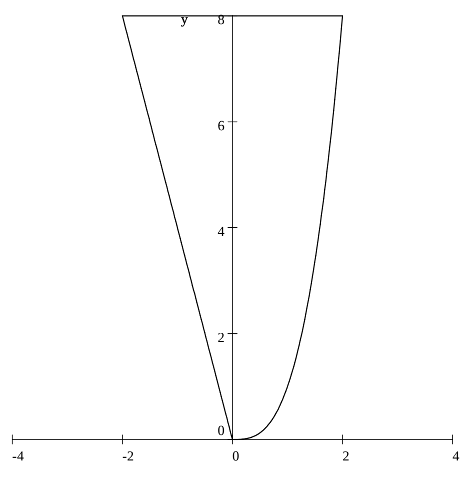
Inner: $\int_0^x (1-2y)\ dy = \left[y - y^2\right]_0^x = x - x^2.$

Outer: $\int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$✓

(#13) Suppose we had the double integral:

$$\int_0^8 \int_{-y/4}^{\sqrt[3]{y}} f(x,y)\ dx\ dy = ???$$

The current order of integration requires us to stripe out the region, horizontally first.



(a) The other curve was $x = -\frac{y}{4} \Rightarrow y = -4x$.

(b) Now explain what must happen if we decide to change the order of integration to $dy\ dx$. Write the appropriate limits of integration.

We see that there are TWO lower functions to deal with... Thus, we will have Integral "A" and Integral "B".

Integral "A":
$y: \quad -4x \to 8$
$x: \quad -2 \to 0$

Integral "B"
$y: \quad x^3 \to 8$
$x: \quad 0 \to 2$

(#14) Suppose we have the upper sheet of the hyperboloid of 2 sheets,

$$-x^2 - y^2 + z^2 = 1.$$

(a) Solve for $z = g(x,y)$. [You only want the positive part!]
$z = g(x,y) = \sqrt{1 + x^2 + y^2}$.

(b) Now suppose we only want the portion of that surface above the unit square $0 \le x \le 1,\ 0 \le y \le 1$.
This will be our S. Suppose the fluid velocity field is $\mathbf{F}(x,y,z) = \langle -z, -z, 3x + 3y \rangle$.
Evaluate the net UPWARD flux integral, where $\mathbf{n}$ is the unit UPWARD normal.

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_R (-Pg_x - Qg_y + R)\, dA$$

We have

$$\begin{aligned} g_x &= \frac{1}{2}\left(1 + x^2 + y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{1 + x^2 + y^2}} \\ g_y &= \frac{y}{\sqrt{1 + x^2 + y^2}}. \end{aligned}$$

The integrand of the transformed double integral is

$$\begin{aligned} -Pg_x - Qg_y + R &= -(-z)\left(\frac{x}{\sqrt{1 + x^2 + y^2}}\right) - (-z)\left(\frac{y}{\sqrt{1 + x^2 + y^2}}\right) + (3x + 3y) \\ &= \sqrt{1 + x^2 + y^2}\left(\frac{x}{\sqrt{1 + x^2 + y^2}}\right) + \sqrt{1 + x^2 + y^2}\left(\frac{y}{\sqrt{1 + x^2 + y^2}}\right) + 3x + 3y \\ &= 4x + 4y. \end{aligned}$$

Thus, the net upward flux integral is

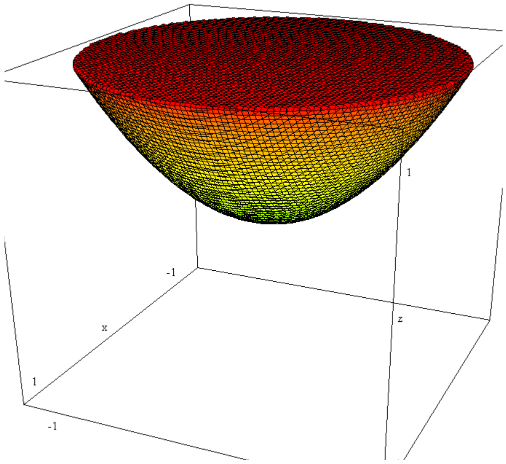
$$\int_0^1 \int_0^1 (4x + 4y)\ dy\ dx = 4.$$

Inner: $\int_0^1 (4x + 4y)\ dy = \left[4xy + 2y^2\right]_{y=0}^1 = 4x + 2.$

Outer: $\int_0^1 (4x + 2)\, dx = \left[2x^2 + 2x\right]_0^1 = 4.$✓

(#15) We have a solid E which bounded by the elliptic paraboloid $z = x^2 + y^2$ $\left[\text{that's } z = r^2\right]$ and the plane $z = 1$.

(a) Make a rough sketch of the solid. The projection down onto the xy-plane is the circle $x^2 + y^2 = 1$.



We have
$z: \quad r^2 \to 1$
$r: \quad 0 \to 1$
$\theta: \quad 0 \to 2\pi.$

(b) Find the outward flux through the surface of E [the closed surface is S] if the vector field is $\mathbf{F}(x,y,z) = \left\langle (1+y)^y, x\sin(x), z^3 \right\rangle$.

$\mathbf{n}$ is the unit OUTWARD normal to S. We apply the Divergence Theorem.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left[(1+y)^y\right] + \frac{\partial}{\partial y}\left[x\sin(x)\right] + \frac{\partial}{\partial z}\left[z^3\right] = 3z^2.$$

$$\oint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \int_0^{2\pi}\int_0^1\int_{r^2}^1 3z^2 \; r \; dz \; dr \; d\theta = \frac{3}{4}\pi.$$

Inner: $r\int_{r^2}^1 3z^2 \; dz = r\left[z^3\right]_{r^2}^1 = r\left(1^3 - \left(r^2\right)^3\right) = r - r^7.$

Next: $\int_0^1 \left(r - r^7\right) dr = \left[\frac{r^2}{2} - \frac{r^8}{8}\right]_0^1 = \frac{3}{8}.$

The Outer integral multiplies the previous result by 2π.