Solutions to Test #1 – MATH 2421
Summer 2005 – T1mikes.tex

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(#1) Find the angle of separation between $\mathbf{u} = \left\langle \sqrt{3},\ 2,\ 3 \right\rangle$ and $\mathbf{v} = \left\langle 1,\ 2\sqrt{3},\ \sqrt{3} \right\rangle$. We have

$$\begin{aligned}\cos(\alpha) &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}| * |\mathbf{v}|} = \frac{\left\langle \sqrt{3},\ 2,\ 3 \right\rangle \cdot \left\langle 1,\ 2\sqrt{3},\ \sqrt{3} \right\rangle}{\sqrt{\left(\sqrt{3}\right)^2 + 2^2 + 3^2} * \sqrt{1^2 + \left(2\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2}} \\ &= \frac{\sqrt{3} + 4\sqrt{3} + 3\sqrt{3}}{\sqrt{3+4+9} * \sqrt{1+12+3}} = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}.\end{aligned}$$

We must have $\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. The correct answer is **(a)**.

(#2) Let us use the standard parameterization for the circle $x^2 + y^2 = 9$,

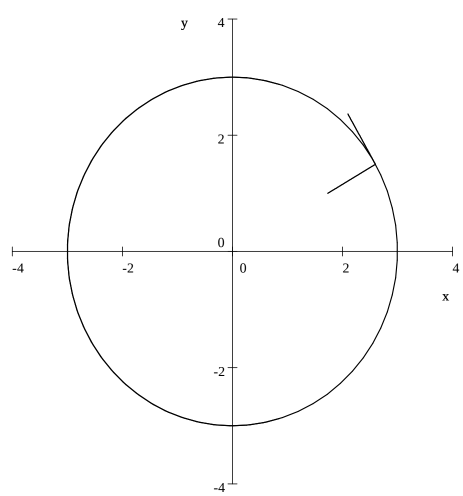
$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle,\ 0 \le t \le 2\pi.$$

What are the values of $\mathbf{T}\left(\frac{\pi}{6}\right)$ and $\mathbf{N}\left(\frac{\pi}{6}\right)$?

Recognize the curve as a circle centered at the origin with radius 3. The orientation of this parameterization is counterclockwise around the circle.

Thus, when $\theta = \frac{\pi}{6}$, the unit tangent vector must be tangent to the circle and point in the counterclockwise direction.

$$\mathbf{v}\left(\frac{\pi}{6}\right) = \left\langle -3\sin\left(\frac{\pi}{6}\right), 3\cos\left(\frac{\pi}{6}\right)\right\rangle = 3\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle \Rightarrow \mathbf{T}(t) = \frac{3\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle}{3\left|\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle\right|} = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle.$$



With a little bit of geometry, we see the direction of $\mathbf{T}$ must be 120°, so

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle.$$

The correct answer must be **(e)**.

Also, recall that whenever we have motion on a circle, the unit normal vector always points back into the center of the circle.

Thus, we must have $\mathbf{N}\left(\frac{\pi}{6}\right) = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$.

(#3) $\langle 2,5,-3\rangle \times \langle -4,0,-7\rangle = ???$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & -3 \\ -4 & 0 & -7 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 5 & -3 \\ 0 & -7 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -3 \\ -4 & -7 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 5 \\ -4 & 0 \end{vmatrix} = \mathbf{i}\,(-35-0) - \mathbf{j}\,(-14-12) + \mathbf{k}\,(0-(-20)) =$$

$-35\mathbf{i} + 26\mathbf{j} + 20\mathbf{k} = \langle -35,\ 26,\ 20\rangle$.

The correct answer is **(d)**.

(#4) If $(x,y,z) = \left(3,\ -\sqrt{3},\ 4\right)$, what is the value of θ in Spherical Coordinates? We only need

$$\tan(\theta) = \frac{y}{x} = -\frac{\sqrt{3}}{3}.$$

Since x is positive and y is negative, our angle must be in Quadrant IV. Thus, we choose

$\theta = \dfrac{11\pi}{6}$. The correct answer is **(d)**.

(#5) If $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and we have a surface $F(x,y,z) = xyz^2 + z - 3 = 0$, then find the value of

$\dfrac{\partial z}{\partial x}$ at $(0,\ 1,\ 3)$.

We have $F_x = yz^2$ and $F_z = 2xyz + 1$.

$$\frac{\partial z}{\partial x} = -\left(\frac{yz^2}{2xyz+1}\right) = -\left(\frac{(1)\left(3^2\right)}{2\,(0)\,(1)\,(3)+1}\right) = -9.$$

The correct answer is **(a)**.

(#6) True or False?

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^{4/3}}{x^2+y^2} = 0.$ FALSE. Convert to polar.

$$\lim_{r\to 0} \frac{(r\cos(\theta))^{4/3}}{r^2} = \lim_{r\to 0} \frac{r^{4/3}(\cos(\theta))^{4/3}}{r^2} = \lim_{r\to 0} \frac{(\cos(\theta))^{4/3}}{r^{2/3}} = d.n.e.$$

The power of r in the numerator was NOT larger than the power in the denominator, and we could not cancel out the r^2 in the denominator. In most directions, the value of

$f(x,y) = \dfrac{x^{4/3}}{x^2+y^2}$ will approach $+\infty$ as $(x,y)\to(0,0)$.

(b) The inequalities $0 \le r \le 5,\ 0 \le z \le 2,\ 0 \le \theta \le 2\pi$ will give us a solid right circular cylinder.
TRUE. In Cylindrical Coordinates, $r = 5$ is a right circular cylinder.

(c) The planes $x + 2y + 3z = 1$ and $9x - 2y - 2z = 1$ are perpendicular to each other.
FALSE. The two normal vectors are $\mathbf{n}_1 = \langle 1,2,3\rangle$ and $\mathbf{n}_2 = \langle 9,-2,-2\rangle$. If the normal vector are orthogonal, then the planes are also orthogonal.

$$\langle 1,2,3\rangle \cdot \langle 9,-2,-2\rangle = 9 - 4 - 6 = -1 \neq 0.$$

The planes are NOT perpendicular to each other.

(#7) Suppose we have the position function $\mathbf{r}(t) = \left\langle t,\ \frac{4\sqrt{2}}{5} * t^{5/4},\ \frac{2}{3}t^{3/2} \right\rangle$.

Find the arc length of the associated 3-D curve from $t = 0$ to $t = 1$.
We need $|\mathbf{r}'(t)|$.

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle 1,\ \frac{4\sqrt{2}}{5}\left(\frac{5}{4}t^{1/4}\right),\ \frac{2}{3}\left(\frac{3}{2}t^{1/2}\right)\right\rangle = \left\langle 1,\ \sqrt{2} * t^{1/4},\ t^{1/2}\right\rangle \\ \left|\mathbf{r}'(t)\right| &= \sqrt{1^2 + \left(\sqrt{2} * t^{1/4}\right)^2 + \left(t^{1/2}\right)^2} = \sqrt{1 + 2\sqrt{t} + t}. \end{aligned}$$

Yes, it's a perfect square. We have $\sqrt{1 + 2\sqrt{t} + t} = \sqrt{\left(1 + \sqrt{t}\right)^2} = 1 + \sqrt{t}$, for $0 \le t \le 1$.
The arc length integral is

$$\int_0^1 \left(1 + t^{1/2}\right) dt = \left[t + \frac{2}{3}t^{3/2}\right]_0^1 = \frac{5}{3}.$$

(#8) A line in 3-D.

$$L_1: \quad \begin{aligned} x &= 6t + 3 \\ y &= -3t + 2 \\ z &= 2t + 1 \end{aligned}$$

(a) What is the Cartesian (Rectangular Coordinates) equation for the plane which is normal to this line at the point $(3, 2, 1)$?
The direction vector is $\mathbf{n} = \langle 6, -3, 2 \rangle$. The standard form is

$$6(x - 3) - 3(y - 2) + 2(z - 1) = 0.$$

(b) Suppose we start at $P(3, 2, 1)$, the base point of our line. We travel along the line until $t = 1$, and we arrive at point Q. Find $\overrightarrow{PQ}$.
When $t = 1$, we have $Q(9, -1, 3)$, and $\overrightarrow{PQ} = \langle 6, -3, 2 \rangle$.

(c) A force vector $\mathbf{F}$ moves an object from P to Q (same as above). We know that $|\mathbf{F}| = 50$ force units and that the angle between the displacement vector and $\mathbf{F}$ is $60°$. Find the amount of work accomplished by $\mathbf{F}$.

$$Work = \mathbf{F} \cdot \overrightarrow{PQ} = |\mathbf{F}| * \left|\overrightarrow{PQ}\right| * \cos(60°) = 50 * 7 * \frac{1}{2} = 175 \text{ work units.}$$

(#9) Surface matching.

(a) (VI) $\phi = \frac{\pi}{6} \Rightarrow$ Upper right circular cone (30° cone).

(b) (I) $x^2 + y^2 - z^2 = 1 \Rightarrow$ Hyperboloid of ONE sheet.

(c) (II) $z = 2\sqrt{r} \Rightarrow$ The θ-plane slice is a square root function.

(d) (IV) $z = y^2 - x^2 \Rightarrow$ Hyperbolic paraboloid (saddle-shape).

(e) (III) $z = -\left(x^2 + y^2\right) \Rightarrow$ Concave down elliptic paraboloid.

(#10) An ant crawls along this surface: $z = f(x, y) = \ln\left(x^2 + y^2 + 1\right)$

(a) Convert this surface equation to Cylindrical Coordinates. This is

$$z = \ln\left(r^2 + 1\right).$$

(b) Is it a surface of revolution about the z-axis?
YES. In Cylindrical Coordinates, the equation does not contain θ. Thus, every θ-plane (rz-plane) trace is identical.

(c) The ant is currently located at $(2, 3, \ln(14))$.
Evaluate $D_{\mathbf{u}} f(2, 3)$ if $\mathbf{u}$ is the unit direction vector associated with $\langle 1, 4\rangle$.

The unit vector is $\mathbf{u} = \dfrac{\langle 1, 4\rangle}{\sqrt{17}}$. We need the gradient vector evaluated at $(2, 3)$.

$$\nabla f = \left\langle \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1} \right\rangle \Rightarrow \nabla f(2, 3) = \left\langle \frac{4}{14}, \frac{6}{14} \right\rangle = \left\langle \frac{2}{7}, \frac{3}{7} \right\rangle.$$

$$D_{\mathbf{u}} f(2, 3) = \left\langle \frac{2}{7}, \frac{3}{7} \right\rangle \cdot \left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle = \frac{2}{7\sqrt{17}} + \frac{12}{7\sqrt{17}} = \frac{14}{7\sqrt{17}} = \frac{2}{\sqrt{17}}.$$

(d) Find a 2-D vector $\langle a, b\rangle$ which tells us in which direction the ant can walk to achieve the greatest positive rate of change in altitude with respect to step size. [In other words, which direction gives us the maximum directional derivative?]
By definition, the gradient vector automatically does this.

We choose $\langle a, b\rangle = \left\langle \dfrac{2}{7}, \dfrac{3}{7} \right\rangle$.

(#11) Here is the position function for projectile motion:

$$\mathbf{r}(t) = \left\langle v_0 \cos(\theta) * t,\ -16t^2 + v_0 \sin(\theta) * t \right\rangle.$$

Tiger Woods hits a golf ball from the origin. The initial speed is $34\sqrt{2}$ ft/sec and the angle of elevation is $45°$. The ball flies directly into the hole!!! The location of the hole is $(x, y) = (???,\ 4)$. With the information given, you should be able to determine precisely how many seconds it took for the ball to land in the hole. We have

$$y(t) = -16t^2 + 34\sqrt{2} * \sin(45°) * t = 4.$$

Thus, we solve

$$\begin{aligned}
-16t^2 + 34\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) t - 4 &= 0 \\
-16t^2 + 34t - 4 &= 0 \\
8t^2 - 17t + 2 &= 0 \\
(8t - 1)(t - 2) &= 0 \Rightarrow t = \frac{1}{8},\ 2.
\end{aligned}$$

Since the hole is *higher* than the initial position, it makes sense that there are TWO times when $y = +4$ feet. On its way up, the ball's height is 4 when $t = \frac{1}{8}$, and then finally, when $t = 2$, it lands in the hole directly!

(#12) Here is a picture of the lever arm vector and a force. (The dashed lines are horizontal and vertical.) The lever arm vector **r** is on the left. Its tail is anchored to the origin. The force vector **F** is applied to the head of the level arm vector. We have $|\mathbf{r}| = 8$ feet and $|\mathbf{F}| = 25$ pounds.

Angle A measures 31°. Angle B measures 28°.

(a) The two dashed lines create a right triangle with the lever arm as its hypotenuse. Since $\angle A = 31°$, the other angle (adjacent to $\angle B$) measures 59°. Thus, the combined angle when the vectors are head-to-head, measures $59° + 28° = 87°$.

When we translate the force vector so that both vectors are tail-to-tail, the angle of separation must also be 87° by alternate interior angles. (The force vectors form parallel lines and the lever arm is the transversal.)

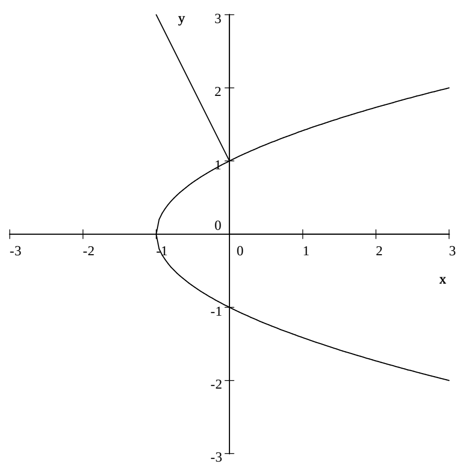
The correct answer is $\alpha = 87°$.

(b) Using the Right-Hand Rule, the shortest angular path from **r** to **F** is counterclockwise. Hence, the torque vector $\boldsymbol{\tau}$ muse point OUT OF THE PAPER.

(c) $|\boldsymbol{\tau}| = |\mathbf{r}| * |\mathbf{F}| * \sin(87°) = 8 * 25 * \sin(87°) = 200 \sin(87°)$ ft-lb of torque.

(#13) Level curve.

(a) Sketch in the level curve for the function $f(x, y) = -x + y^2 - 1$ when $k = 0$.



The level curve is
$-x + y^2 - 1 = 0 \Rightarrow$
$x = y^2 - 1$.

This is a sideways parabola which opens outward to the right.

(b) Sketch in the gradient vector at the point $(0, 1)$ on the y-axis. In this case, be sure that the gradient vector IS drawn exactly to scale, because it *will* fit!

$$\nabla f = \langle -1, \ 2y \rangle \Rightarrow \nabla f = \langle -1, 2 \rangle .$$

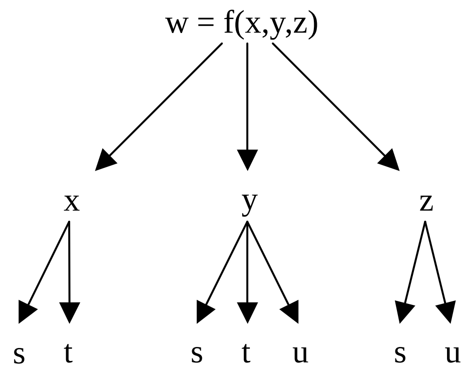
(c) Explain the geometric relationship between the gradient vector and the level curve.

The gradient vector is always normal to the level curve and it points to the next higher level curve. If we take a step in that direction, we expect to gain altitude.

(#14) Multivariable Chain Rule...

We have $w(s,t,u) = f(x(s,t), y(s,t,u), z(s,u))$ after appropriate compositions on $w = f(x,y,z)$.

(a) Draw an appropriate tree diagram for w.



(b) Suppose that second-level functions are

$$\begin{aligned} x(s,t) &= \sqrt{s^2+t^2} \\ y(s,t,u) &= \sqrt{s^2+t^2+u^2} \\ z(s,u) &= e^{2su} \end{aligned}$$

and we do not have a specific formula for $f(x,y,z)$.

Find an algebraic expression for $\dfrac{\partial w}{\partial u}$ and simplify as much as possible.

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

The function for x doesn't have any u's in it, so $\dfrac{\partial x}{\partial u} = 0$.

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial y}\left(\frac{u}{\sqrt{s^2+t^2+u^2}}\right) + \frac{\partial f}{\partial z}\left(2se^{2su}\right).$$

(#15) The kinetic energy of a mass is $E = \dfrac{1}{2}mv^2$, where m is the mass in kilograms and v is the velocity in meters per second.

Use the total differential to estimate largest (positive) error in E if our measurements are $m = 10 \pm 0.1$ kg and $v = 6 \pm 0.2$ m/sec.

$$\begin{aligned} dE &= E_m dm + E_v dv \\ &= \frac{1}{2}v^2\ dm + mv\ dv \\ &= \frac{1}{2}\left(6^2\right)\ dm + (10)(6)\ dv \\ &= 18\ dm + 60\ dv. \end{aligned}$$

We choose $dm = +0.1$ and $dv = +0.2$.

$dE = 18\,(0.1) + 60\,(0.2) = 13.8$. Thus, our calculation of energy is

$$E = \frac{1}{2}\,(10)\,\left(6^2\right) = 180 \pm 13.8 \text{ energy units.}$$

(#16) We have two surfaces. Both are right circular cylinders.

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 + z^2 &= 4 \end{aligned}$$

Their intersection is a 3-D curve. Find an appropriate parameterization for the 3-D curve.

(a) We have two ellipses in 3-D which perpendicular to each other. They are both perpendicular to the yz-plane.

(b) We start with

$$\begin{aligned} x &= 2\cos(t) \\ y &= 2\sin(t) \end{aligned}$$

Since $z^2 = 4 - x^2 \Rightarrow z = \pm\sqrt{4 - x^2} = \pm\sqrt{4 - 4\cos^2(t)} = \pm\sqrt{4\sin^2(t)} = \pm 2\sin(t)$.
This gives us TWO parameterizations.
$\mathbf{r}_1(t) = \langle 2\cos(t),\ 2\sin(t),\ -2\sin(t)\rangle$
$\mathbf{r}_{21}(t) = \langle 2\cos(t),\ 2\sin(t),\ 2\sin(t)\rangle$.
Both domains are $0 \le t \le 2\pi$.
The first one is contained in the plane $z = -y$ and the second one is contained in $z = y$. Hence, both curves are perpendicular to the yz-plane.