Note Sheet for Calculus III – MATH 2421

Summer 2006

1. 2D & 3D Vectors

   $\mathbf{a} = \langle a_1,\ a_2,\ a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

   The unit direction vector associated with nonzero $\mathbf{a}$ is $\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$.

   $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| * \|\mathbf{b}\| * \cos(\alpha)$. The angle of separation is α.

   $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. $\quad \mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a}$ and $\mathbf{b}$ are orthogonal.

   $\mathbf{Proj_b a} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \right) \left(\dfrac{\mathbf{b}}{\|\mathbf{b}\|} \right) = (comp_{\mathbf{b}}\mathbf{a}) \left(\dfrac{\mathbf{b}}{\|\mathbf{b}\|} \right)$.

   The scalar component is the directed length of $\mathbf{Proj_b a}$.

   $Work = \mathbf{F} \cdot \overrightarrow{PQ}$.

   $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| * \|\mathbf{b}\| * \sin(\alpha) =$ Area of parallelogram.

   $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. [Use the Right-Hand Rule.]

2. Lines and Surfaces

   If a 3D line passes through $P(x_1,\ y_1,\ z_1)$ with a direction vector $\mathbf{v} = \langle a, b, c \rangle$, then the parametric form for the line is:

   $$\begin{aligned} x &= (a)\,t + x_1 \\ y &= (b)\,t + y_1 \\ z &= (c)\,t + z_1 \end{aligned}$$

   Standard form for a plane with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

   $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

   A cylindrical surrface with only x and y represented is perpendicular to the xy-plane.

   Quadric Surface examples:

   $\dfrac{x^2}{1^2} + \dfrac{y^2}{2^2} + \dfrac{z^2}{3^2} = 1$. Ellipsoid. $\qquad z = \dfrac{x^2}{1^2} + \dfrac{y^2}{2^2}$. Elliptic paraboloid.

   $z = \dfrac{x^2}{1^2} - \dfrac{y^2}{2^2}$. Hyperbolic paraboloid (saddle). $\qquad z = \sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$. Upper elliptic cone.

   $\dfrac{x^2}{1^2} + \dfrac{y^2}{2^2} - \dfrac{z^2}{3^2} = 1$. Hyperboloid of ONE sheet. (One negative sign.)

   $-\dfrac{x^2}{1^2} - \dfrac{y^2}{2^2} + \dfrac{z^2}{3^2} = 1$. Hyperboloid of TWO sheets. (Two negative signs.)

3. Vector-valued Functions and Arc Length

   The parameterization for $y = x^2$ is $\mathbf{r}(t) = \left\langle t,\ t^2 \right\rangle,\ t \in \mathbb{R}$.

   $\mathbf{v}(t) = \mathbf{r}'(t) \qquad \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

   If the position function is $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$, then

$$ds = \|\mathbf{v}(t)\|\, dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

   $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|},\ \mathbf{v}(t) \neq \mathbf{0}. \qquad \mathbf{T}(t)$ has the same direction as $\mathbf{v}(t)$.

   $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},\ \mathbf{T}'(t) \neq \mathbf{0}. \qquad \mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$.

4. Multivariable Functions and Partial Derivatives

   $z = f(x, y)$ is a 3D surface and $f(x, y) = 0$ is a level curve of f.

   For different values of k, when the level curves of f appear to cross at $(x_0,\ y_0)$, then f is discontinuous at $(x_0,\ y_0)$.

   Total differential: $dz = \left(\dfrac{\partial f}{\partial x}\right) dx + \left(\dfrac{\partial f}{\partial y}\right) dy = f_x dx + f_y dy$.

   Chain Rule (Related Rates): $\dfrac{dz}{dt} = \left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{dx}{dt}\right) + \left(\dfrac{\partial z}{\partial y}\right)\left(\dfrac{dy}{dt}\right)$.

   $\nabla f(x, y, z) = \langle f_x,\ f_y,\ f_z \rangle$. The gradient vector always points in the direction of the greatest directional derivative (best rate of increase in f).

   If $w = f(x, y, z)$, then $\nabla f(x, y, z) = \langle f_x,\ f_y,\ f_z \rangle$.

   If $\mathbf{u}$ is a unit vector, then the directional derivative of f in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

   [This is the slope of a real honest-to-goodness tangent line in some plane.]

   In order to find the tangent plane to a surface $z = f(x, y)$, we create the parent function

$$F(x, y, z) = z - f(x, y).$$

   ∇F is always normal to the level surface.