Solutions to Quiz #06 – MATH 2421
Summer 2006

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(#1) Spherical coordinates.

(a) If $(\rho, \theta, \phi) = \left(4,\ \frac{\pi}{6},\ \frac{2\pi}{3}\right)$, then find the associated value of x.

Since $x = r\cos(\theta)$ and $r = \rho\sin(\phi)$, we must have

$$x = \rho\sin(\phi)\cos(\theta) = 4\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 3.$$

(b) If $(x, y, z) = (4,\ -4,\ 2)$, then find the associated value of θ.
Since $x > 0$ and $y < 0$, θ must be Quadrant IV.

$$\tan(\theta) = \frac{y}{x} = \frac{(-4)}{4} = -1.$$

We have $\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$ $\left(\text{or } \frac{7\pi}{4}\right)$.

(#2) Find the mass integral if $\sigma(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and the solid is
$0 \le \rho \le \sin(\phi),\ 0 \le \phi \le \pi$.
DO NOT EVALUATE. [It's too long for a quiz.]
$dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$.

The integrand becomes $\sqrt{x^2 + y^2 + z^2} \Rightarrow \rho$.
I've already given most of the information for the limits of integration.
$\rho:\ \ 0 \to \sin(\phi)$
$\phi:\ \ 0 \to \pi$
$\theta:\ \ 0 \to 2\pi$.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin(\phi)} \rho * \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta = \frac{8\pi}{15} \text{ mass units.}$$

We note that the Inner integral would have been

$$\sin(\phi)\int_0^{\sin(\phi)} \rho^3\ d\rho = \frac{\sin^5(\phi)}{4}.$$

This would have been ugly, but not impossible.

$$\begin{aligned}\sin^5(\phi) &= \left(1 - \cos^2(\phi)\right)^2\sin(\phi)\\ &= \left(\cos^4(\phi) - 2\cos^2(\phi) + 1\right)\sin(\phi)\\ &= \cos^4(\phi)\sin(\phi) - 2\cos^2(\phi)\sin(\phi) + \sin(\phi).\end{aligned}$$

The first two products require the substitution $u = \cos(\phi),\ du = -\sin(\phi)\,d\phi$.