Solutions to Quiz #02 – MATH 2421
Summer 2006

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1. Dot product.

   (a) A force vector $\mathbf{F} = \langle 30,\ -20\rangle$ pounds pushes an object from $P\,(0,\ 40)$ to $Q\,(10,\ 20)$ [measured in feet].
   Find the work accomplished by $\mathbf{F}$.
   First, find the displacement vector $\overrightarrow{PQ}$.

   $$\overrightarrow{PQ} = \langle 10-0,\ 20-40\rangle = \langle 10,\ -20\rangle\,.$$

   The dot product represents the work accomplished by $\mathbf{F}$.

   $$\mathbf{F}\cdot\overrightarrow{PQ} = \langle 30,\ -20\rangle\cdot\langle 10,\ -20\rangle = (30)\,(10) + (-20)\,(-20) = 700 \text{ work units.}$$

   (b) $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\| * \|\mathbf{b}\| * \cos(\alpha)$
   Find the angle of separation between $\mathbf{a} = \langle 1,\ 2,\ -2\rangle$ and $\mathbf{b} = \langle -2,\ 4,\ 3\rangle$.

   $$\mathbf{a}\cdot\mathbf{b} = \langle 1,\ 2,\ -2\rangle\cdot\langle -2,\ 4,\ 3\rangle = -2+8-6 = 0.$$

   We recall that

   $$\cos(\alpha) = 0 \Rightarrow \alpha = \cos^{-1}(0) = \frac{\pi}{2}.$$

   The vectors are orthogonal.

2. Cross product.

   (a) Torque. The point of rotation is located at P. Angle A measures 30°. The force is applied perpendicular to the pedal, and the pedal is horizontal.
   The magnitude of the lever arm vector (from P) is one foot. The magnitude of the force is 60 pounds.

   $$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\| * \|\mathbf{b}\| * \sin(\alpha)$$

   If we translate the force vector so that its tail coincides with the tail of the lever arm vector, then the angle of separation is $90^\circ + 30^\circ = 120^\circ$.
   The shortest angular path from the lever arm vector to the force vector is *clockwise.*

   (i) In what direction does the torque vector $(\mathbf{r}\times\mathbf{F})$ point?
   The torque vector points INTO the paper (clockwise).

   (ii) What is the magnitude of $\mathbf{r}\times\mathbf{F}$?

   $$\|\mathbf{r}\times\mathbf{F}\| = \|\mathbf{r}\| * \|\mathbf{F}\| * \sin(\alpha) = (1 \text{ ft})\,(60 \text{ lb})\sin(120^\circ) = 30\sqrt{3} \text{ ft-lb of torque.}$$

(b) Evaluate this cross product!

$$\begin{aligned}
\langle -3,0,4\rangle \times \langle 2,-5,1\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & 4 \\ 2 & -5 & 1 \end{vmatrix} = \\
&= \mathbf{i}\begin{vmatrix} 0 & 4 \\ -5 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -3 & 4 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -3 & 0 \\ 2 & -5 \end{vmatrix} \\
&= \mathbf{i}\,(0-(-20)) - \mathbf{j}\,(-3-8) + \mathbf{k}\,(15-0) \\
&= 20\mathbf{i} + 11\mathbf{j} + 15\mathbf{k} = \langle 20,\ 11,\ 15\rangle\,.
\end{aligned}$$