Solutions to Assignment #11 – MATH 2421
Spring 2006

Kawai

Section 13.3

(#18) This is a conservative field. $\mathbf{F}(x,y,z) = \langle e^y,\ xe^y,\ (z+1)e^z \rangle$

We can check this using $\nabla \times \mathbf{F} = \mathbf{0}$.

$$f(x,y,z) = \int P\,dx = \int Q\,dy = \int R\,dz.$$

Any terms which have $x,\ y$, AND z must be identical in each of the three integrals. Otherwise, we just pick one term of each type when we scan the three results. The sum is the potential function $f(x,y,z)$.

$$\int P\,dx = \int e^y\,dx = xe^y + \alpha(y,z).$$

Since the variable of integration is x, any term which has y's or z's (or both) will be considered constant with respect to x. Hence, the symbol $\alpha(y,z)$ represent the "constant of integration" for this integral.

$$\int Q\,dy = \int xe^y\,dy = xe^y + \beta(x,z).$$

This is good. The xe^y term matched up in both integrals.

$$\int R\,dz = \int (z+1)e^z = ze^z + \gamma(x,y).$$

This last one requires integration by parts!

There are NO terms with $x,\ y$, AND z. We collect one copy of each of the other terms when we form the potential function.

$$f(x,y,z) = xe^y + ze^z + C.$$

You should verify that $\nabla f = \mathbf{F}$.

The Fundamental Theorem of Line Integration tells us that the value of any work integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path.*

As long as $\mathbf{F}$ is defined on the path, the value of the integral is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(end) - f(start).$$

In this case, the path defined by C is

$$\mathbf{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle,\ 0 \le t \le 1.$$

Thus, the starting point is $(0,0,0)$ and the ending point is $(1,1,1)$.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(1,1,1) - f(0,0,0)\\ &= \left(1e^1 + 1e^1\right) - \left(0e^0 + 0e^0\right) = 2e \text{ work units.}\end{aligned}$$

Let's compare this with the parameterized line integral.

$$d\mathbf{r} = \left\langle 1,\ 2t,\ 3t^2 \right\rangle dt$$

$$\mathbf{F} = \left\langle e^{t^2},\ te^{t^2},\ \left(t^3+1\right) e^{t^3} \right\rangle$$

The work integral is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left\langle e^{t^2},\ te^{t^2},\ \left(t^3+1\right) e^{t^3} \right\rangle \cdot \left\langle 1,\ 2t,\ 3t^2 \right\rangle dt \\ &= \int_0^1 \left(e^{t^2} + 2t^2 e^{t^2} + \left(3t^5 + 3t^2\right) e^{t^3} \right) dt\end{aligned}$$

Interesting observation: We know that $\int e^{t^2} dt$ does NOT have an elementary antiderivative. So this looks like the evaluation might be impossible. Not so!

Since the second term, $\int 2t^2 e^{t^2} dt$, requires integration by parts:

$$\int 2t^2 e^{t^2} dt = te^{t^2} - \int e^{t^2} dt,$$

the first integral will be cancelled!

Thus, we see that the algebra can be performed, but we will still hate the tricks involved. Aren't we glad that we have the potential function?!

(#20) When we have two-variable field $\mathbf{F} = \langle P, Q\rangle$, the test for conservative field is simple. We simply check to see if

$$Q_x - P_y = 0.$$

We have

$$\begin{aligned} P &= 2y^2 - 12x^3y^3 \\ Q &= 4xy - 9x^4y^2 \end{aligned}$$

and

$$Q_x - P_y = 4y - 36x^3y^2 - \left(4y - 36x^3y^2\right) = 0.\checkmark$$

We create the potential function.

$$\begin{aligned} \int P dx &= \int \left(2y^2 - 12x^3y^3\right) dx = 2xy^2 - 3x^4y^3 + \alpha(y) \\ \int Q dy &= \int \left(4xy - 9x^4y^2\right) dy = 2xy^2 - 3x^4y^3 + \beta(x). \end{aligned}$$

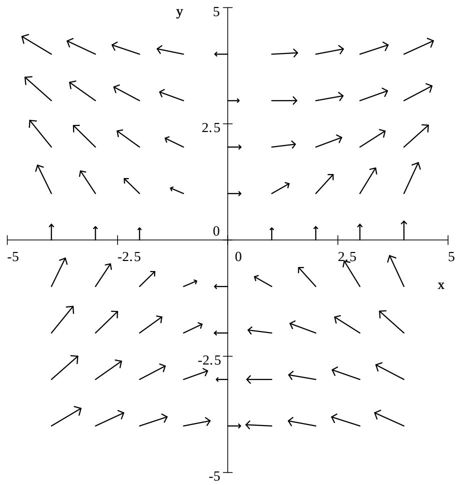
The terms which contain both variables must be identical. Thus, we have

$$f(x,y) = 2xy^2 - 3x^4y^3 + C.$$

We choose any path from $(1,1)$ to $(3,2)$.

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= f(3,2) - f(1,1) \\ &= 2(3)\left(2^2\right) - 3\left(3^4\right)\left(2^3\right) - (2-3) = -1919 \text{ work units.} \end{aligned}$$

(#24) Here's a plot of $\mathbf{F}(x,y) = \langle 2xy + \sin(y),\ x^2 + x\cos(y) \rangle$.



It looks reasonably *irrotational.*
That is, it doesn't look like there are any closed paths which have a nonzero circulation (the arrow go around in a circle).

We check $Q_x - P_y$.

$Q_x = 2x + \cos(y)$
$P_y = 2x + \cos(y)$. They are equal, so $Q_x - P_y = 0$.

F is definitely conservative!

Section 13.4

(#10) The integral $\oint_C \left(y^2 - \tan^{-1}(x)\right) dx + (3x + \sin(y))\, dy = \oint_C P dx + Q dy$ is the same as

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

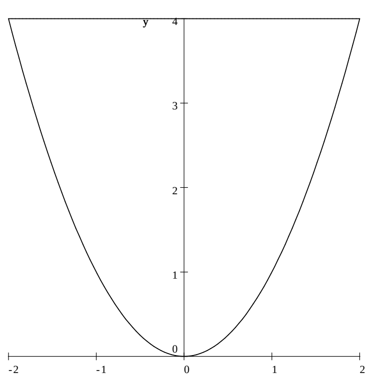
where $\mathbf{F} = \langle P, Q \rangle$. This is a circulation integral. [If **F** is a conservative field, then this is automatically zero.]

We calculate the circulation density

$$\begin{aligned} Q_x - P_y &= \frac{\partial}{\partial x}[3x + \sin(y)] + \frac{\partial}{\partial y}\left[y^2 - \tan^{-1}(x)\right] \\ &= 3 + 2y. \end{aligned}$$

We integrate this over the *interior* of C. This is certainly easier to do than evaluating two line integrals (two pieces) around the boundary C. In fact, let's see what those two integrals would look like...

Here's the region. We're bounded by $y = x^2$ and $y = 4$.



For the parabolic edge, we use
$\mathbf{r}(t) = \langle t,\ t^2 \rangle$, $-2 \le t \le 2$.
$d\mathbf{r} = \langle 1,\ 2t \rangle\, dt$.

We evaluate **F** on that edge:

$$\mathbf{F} = \left\langle \left(t^2\right)^2 - \tan^{-1}(t),\ 3t + \sin\left(t^2\right) \right\rangle$$

$$= \left\langle t^4 - \tan^{-1}(t),\ 3t + \sin\left(t^2\right) \right\rangle.$$

The circulation on that edge is (using the counterclockwise orientation)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-2}^{2} \left\langle t^4 - \tan^{-1}(t),\ 3t+\sin\left(t^2\right)\right\rangle \cdot \langle 1,\ 2t\rangle\, dt \\ &= \int_{-2}^{2} \left(t^4 - \tan^{-1}(t) + 6t^2 + 2t\sin\left(t^2\right)\right) dt\end{aligned}$$

Each of these functions has an elementary antiderivative, but, for example, $\tan^{-1}(t)$ requires integration by parts (a lot of work).

For the upper (horizontal) edge, we can find the integral over the line from $(-2,0)$ to $(2,0)$, and then take the negative of the integral, because we originally wanted the value when the orientation is in the opposite direction (from right to left is counterclockwise over C).

Since the line integral is

$$\int_C Pdx + Qdy$$

and y is not changing (it's equal to 4 at all points of the line segment), we have $dy = 0$ and

$$\int_C Pdx + Qdy = \int_C Pdx = \int_{-2}^{2} \left(4^2 - \tan^{-1}(x)\right) dx.$$

Now we see that the

$$\int_{-2}^{2} \tan^{-1}(x)\ dx$$

in both integrals will cancel! Green's Theorem will automatically discard offsetting integrals like this!

We will count the area R (interior of S) vertically first.

$dy\ dx$

$y:\quad x^2 \to 4$

$x:\quad -2 \to 2.$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2}\int_{x^2}^{4} (3+2y)\ dy\ dx = \frac{416}{5}.$$

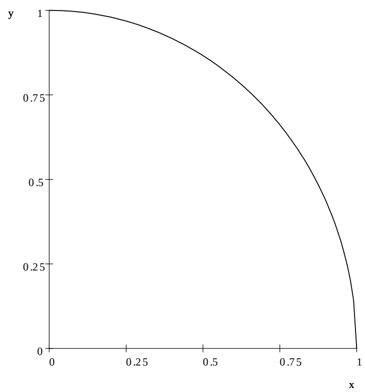
Inner:

$$\int_{x^2}^{4} (3+2y)\ dy = \left[3y+y^2\right]_{x^2}^{4} = (12+16) - \left(3x^2 + x^4\right) = -3x^2 - x^4 + 28.$$

Outer:

$$\int_{-2}^{2} \left(-3x^2 - x^4 + 28\right) dx = \left[-x^3 - \frac{x^5}{5} + 28x\right]_{-2}^{2} = \frac{416}{5}.\checkmark$$

(#17) Here's the simple closed path C (once around counterclockwise).



The enclosed region is certainly polar friendly.

$r\ dr\ d\theta$
$r: \quad 0 \to 1$
$\theta: \quad 0 \to \pi/2.$

The integrand becomes

$$Q_x - P_y = \frac{\partial}{\partial x}\left[xy^2\right] - \frac{\partial}{\partial y}\left[x^2 + xy\right] = y^2 - x.$$

In polar, this is $r^2 \sin^2(\theta) - r\cos(\theta)$.

Green's Theorem give us

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \int_0^1 \left(r^2 \sin^2(\theta) - r\cos(\theta)\right) r\ dr\ d\theta = \frac{\pi}{16} - \frac{1}{3}.$$

Inner:

$$\begin{aligned} \int_0^1 \left(r^3 \sin^2(\theta) - r^2 \cos(\theta)\right)\ dr &= \sin^2(\theta) \int_0^1 r^3 dr - \cos(\theta) \int_0^1 r^2 dr \\ &= \frac{1}{4}\sin^2(\theta) - \frac{1}{3}\cos(\theta). \end{aligned}$$

Outer:

$$\begin{aligned} \frac{1}{4}\int_0^{\pi/2} \sin^2(\theta)\, d\theta - \frac{1}{3}\int_0^{\pi/2} \cos(\theta)\, d\theta &= \frac{1}{4}\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} - \frac{1}{3}\left[\sin(\theta)\right]_0^{\pi/2} \\ &= \frac{\pi}{16} - \frac{1}{3}.\checkmark \end{aligned}$$

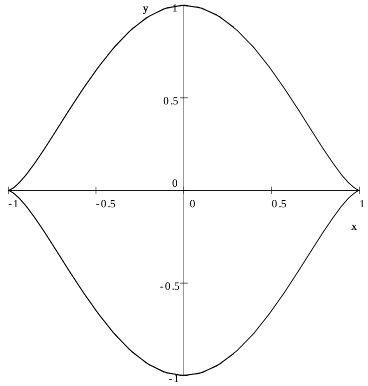
(#20) The area formula on p. 948 says that area enclosed by C is

$$A = \oint_C x\ dy = -\oint_C y\ dx = \frac{1}{2}\oint_C x\ dy - y\ dx.$$

Why does this work? We have $P = -y$ and $Q = x$, and thus, by Green's Theorem, we have

$$\frac{1}{2}\oint_C x\ dy - y\ dx = \frac{1}{2}\iint_R (1 - (-1))\, dA = \iint_R 1\ dA = Area\ of\ R.$$

Our problem has the curve $\mathbf{r}(t) = \left\langle \cos(t),\ \sin^3(t) \right\rangle,\ 0 \le t \le 2\pi.$



It turns out that the double integral is somewhat difficult to evaluate since

$$x = \cos(t) \text{ and } y = \sin^3(t) = \left(\sqrt{1-\cos^2(t)}\right)^3$$

The explicit form of the curve is

$$y = \left(1 - x^2\right)^{3/2}.$$

The basic single-variable integral

$$\int_0^1 \left(1-x^2\right)^{3/2} = \frac{3\pi}{16}$$

equal one-fourth of the enclosed area. It can be evaluated using a trigonometric substitution. We will show that the line integral is easier to evaluate this time.

$$\begin{aligned} A &= \oint_C x\ dy = \int_0^{2\pi} \cos(t)\left(3\sin^2(t)\cos(t)\,dt\right) \\ &= 3\int_0^{2\pi} \sin^2(t)\cos^2(t)\,dt = 3\int_0^{2\pi}\left(\frac{\sin(2t)}{2}\right)^2 dt \\ &= \frac{3}{4}\int_0^{2\pi}\sin^2(2t)\,dt = \frac{3}{4}\int_0^{2\pi}\left(\frac{1}{2}-\frac{\cos(4t)}{2}\right)dt \\ &= \frac{3}{4}\left[\frac{t}{2}-\frac{\sin(4t)}{8}\right]_0^{2\pi} = \frac{3\pi}{4}.\checkmark \end{aligned}$$

(#26) The moment of inertia formulas are given in Problem #25:

$$I_y = \left(\frac{\sigma}{3}\right)\oint_C x^3 dy$$

Our path C is a circle (centered at the origin) of radius a.

$$\begin{aligned} x &= a\cos(t) \\ y &= a\sin(t),\ 0 \le t \le 2\pi. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_y &= \left(\frac{\sigma}{3}\right)\int_0^{2\pi} a^3\cos^3(t)\left(a\cos(t)\,dt\right) \\ &= \left(\frac{a^4\sigma}{3}\right)\int_0^{2\pi}\cos^4(t)\,dt = \left(\frac{a^4\sigma}{3}\right)\left(\frac{3\pi}{4}\right) = \frac{a^4\pi}{4}. \end{aligned}$$

We note that $\cos^4(t)$ requires the "reduction of power" formula twice.

Section 13.8

(#12) Divergence Theorem. Cylindrical Coordinates.

We have $\mathbf{F} = \left\langle x^3,\ 2xz^2,\ 3y^2z \right\rangle$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left[x^3\right] + \frac{\partial}{\partial y}\left[2xz^2\right] + \frac{\partial}{\partial z}\left[3y^2z\right] = 3x^2 + 0 + 3y^2 \Rightarrow 3r^2.$$

The outward flux is equal to the flux density integrated over the interior of S.

The lower surface is $z = 0$ and the upper surface is the elliptic paraboloid, $z = 4 - x^2 - y^2 = 4 - r^2$.

If we smash this down onto the xy-plane, we have the circle $x^2 + y^2 \leq 4$.

$dz\ r\ dr\ d\theta$

$z:\quad 0 \rightarrow 4 - r^2$

$r:\quad 0 \rightarrow 2$

$\theta:\quad 0 \rightarrow 2\pi$.

$$\oint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^2\ r\ dz\ dr\ d\theta = +32\pi.$$

The outward flux is positive through S.

Inner:

$$3r^3 \int_0^{4-r^2} dz = 3r^3\left(4 - r^2\right) = 12r^3 - 3r^5.$$

Next:

$$\int_0^2 \left(12r^3 - 3r^5\right) dr = \left[3r^4 - \frac{r^6}{2}\right]_0^2 = 48 - 32 = 16.$$

The Outer integral multiplies the previous result by 2π. The final answer is 32π.✓

(#13) Divergence Theorem. Spherical Coordinates.

We have $\mathbf{F} = \left\langle x^3,\ y^3,\ z^3 \right\rangle$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left[x^3\right] + \frac{\partial}{\partial y}\left[y^3\right] + \frac{\partial}{\partial z}\left[z^3\right] = 3x^2 + 3y^2 + 3z^2 = 3\rho^2.$$

The outward flux is equal to the flux density integrated over the interior of S.

$\rho^2 \sin(\phi)\ \ d\rho\ d\phi\ d\theta$

$\rho:\quad 0 \rightarrow 1$

$\phi:\quad 0 \rightarrow \pi$

$\theta:\quad 0 \rightarrow 2\pi$.

$$\oint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^2\ \rho^2 \sin(\phi)\ \ d\rho\ d\phi\ d\theta = \frac{12\pi}{5}.$$

Inner:

$$3\sin(\phi)\int_0^1 \rho^4 d\rho = \frac{3}{5}\sin(\phi).$$

Next:

$$\frac{3}{5}\int_0^\pi \sin(\phi)\,d\phi = -\frac{3}{5}\left[\cos(\phi)\right]_0^\pi = -\frac{3}{5}(-1-1) = \frac{6}{5}.$$

The Outer integral multiplies the previous result by 2π. The final answer is $\frac{12\pi}{5}$.✓