Some Solutions to Homework #01 – MATH 2421
Spring 2006

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Section 9.1

(#14) Find the center and radius of this sphere.

Hint: Divide everything by 4 first.

$$\begin{aligned} 4x^2 - 8x + 4y^2 + 16y + 4z^2 &= 1 \\ x^2 - 2x + y^2 + 4y + z^2 &= \frac{1}{4}. \end{aligned}$$

NOW complete the square twice.

$$\begin{aligned} x^2 - 2x + (-1)^2 + y^2 + 4y + \left(2^2\right) + z^2 &= \frac{1}{4} + 1 + 4 \\ (x-1)^2 + (y+2)^2 + (z-0)^2 &= \frac{21}{4} = R^2. \end{aligned}$$

The center is located at $C\,(1, -2, 0)$ and the radius is $R = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

(#16) Use the Midpoint formula from Problem #15 to find the coordinates of the center of the sphere.

$C\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = C\,(3, 2, 7)\,.$

The length of the diameter is

$$\sqrt{(4-2)^2 + (3-1)^2 + (10-4)^2} = \sqrt{44} = 2\sqrt{11}.$$

The radius has length $R = \sqrt{11}$. The standard form for the sphere is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = \left(\sqrt{11}\right)^2 = 11.$$

(#20) The equation $x = 10$ is the plane parallel to the yz-plane which is 10 units in *front* of that plane.

(#22) The inequality $y \geq 0$ is the region to the *right* of the xz-plane and including that plane.

This also includes Octants I, II, V, and VI.

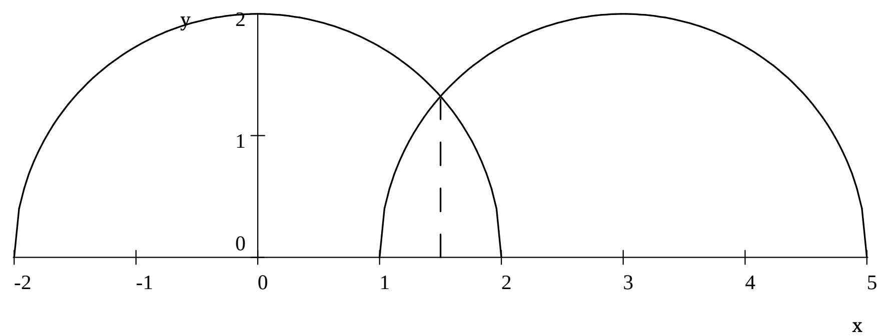
(#26) The inequality $1 \leq x^2 + y^2 + z^2 \leq 25$ is the region between the spheres of radius 1 and 5, including those spheres.

These spheres are centered at the origin.

(#36) The details for transforming the problem to two spheres of radius 2 whose centers are 3 units apart were given in class.

The disc method gives us one-half of the intersecting volume.

[Recall that the equation of the left circle is $x^2 + y^2 = 4 \Rightarrow y = \sqrt{4 - x^2}$. We form the solid of revolution based on the diagram below. Our region is in the intersection, and we want the right half of it. This will be revolved about the horizontal axis!]



$$\pi \int_{x_1}^{x_2} y^2 dx = \pi \int_{3/2}^{2} \left(\sqrt{4 - x^2}\right)^2 dx = \pi \int_{3/2}^{2} \left(4 - x^2\right) dx = \frac{11\pi}{24}.$$

The total volume is $\dfrac{11\pi}{12}$.

Section 9.2

(#4) We need head-to-tail combinations. Remember that it's "first tail to last head".

(a) $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. The first tail is P. The last head is R.

(b) $\overrightarrow{RP} + \overrightarrow{PS} = \overrightarrow{RS}$.

(c) $\overrightarrow{QS} - \overrightarrow{PS} =???$ This must be the vector which connects two heads!
We know that $\overrightarrow{QP} + \overrightarrow{PS} = \overrightarrow{QS}$, so by algebra, we have

$$\overrightarrow{QP} = \overrightarrow{QS} - \overrightarrow{PS}.$$

(d) $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ} = \overrightarrow{RQ}$.

(#18) We have $\mathbf{a} = \langle 3, 0, -2\rangle$ and $\mathbf{b} = \langle 1, -1, 1\rangle$.

(a) $|\langle 3, 0, -2\rangle| = \sqrt{3^2 + 0^2 + (-2)^2} = \sqrt{13}.$

(b) $\langle 3, 0, -2\rangle + \langle 1, -1, 1\rangle = \langle 4, -1, -1\rangle$.

(c) $\langle 3, 0, -2\rangle - \langle 1, -1, 1\rangle = \langle 2, 1, -3\rangle$.

(d) $2\langle 3, 0, -2\rangle = \langle 6, 0, -4\rangle$.

(e) $3\langle 3, 0, -2\rangle + 4\langle 1, -1, 1\rangle = \langle 9, 0, -6\rangle + \langle 4, -4, 4\rangle = \langle 13, -4, -2\rangle$.

(#19) The associated unit vector is $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \dfrac{\langle 8, -1, 4\rangle}{\sqrt{8^2 + (-1)^2 + 4^2}} = \dfrac{1}{9}\langle 8, -1, 4\rangle = \left\langle \dfrac{8}{9}, -\dfrac{1}{9}, \dfrac{4}{9}\right\rangle$.

(#20) Find a vector that has the same direction as $\langle -2, 4, 2\rangle$, but has length 6.

First, form the associated unit direction vector by dividing the original vector by its magnitude.

$|\langle -2, 4, 2\rangle| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}.$

The unit direction vector is $\dfrac{1}{2\sqrt{6}}\langle -2, 4, 2\rangle = \left\langle -\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right\rangle$.

$\left(\text{If we rationalize the denominators, it actually looks like this: } \left\langle -\dfrac{\sqrt{6}}{6}, \dfrac{\sqrt{6}}{3}, \dfrac{\sqrt{6}}{6}\right\rangle.\right)$

We must now stretch this vector out to length 6 by multiplying this unit vector by 6.

$$6\left\langle -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right\rangle = \left\langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6}\right\rangle \quad \text{or} \quad \frac{6}{2\sqrt{6}}\langle -2, 4, 2\rangle = \frac{\sqrt{6}}{2}\langle -2, 4, 2\rangle.$$

(#22) If $\theta = 38°$ is in standard position (assume that this is so), then the component form of the force vector must be

$$|\mathbf{F}|\langle \cos(\theta), \sin(\theta)\rangle = \langle 50\cos(38°),\ 50\sin(38°)\rangle.$$

The horizontal component is $50\cos(38°)\mathbf{i}$ and the vertical component is $50\sin(38°)\mathbf{j}$.

Section 9.3

(#8) $\langle 0, 4, -3\rangle \cdot \langle 2, 4, 6\rangle = (0)(2) + (4)(4) + (-3)(6) = -2.$

[We note that α must be obtuse.]

(#14) $\cos(\alpha) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}| * |\mathbf{b}|} = \dfrac{\langle 6, -3, 2\rangle \cdot \langle 2, 1, -2\rangle}{|\langle 6, -3, 2\rangle| \cdot |\langle 2, 1, -2\rangle|} = \dfrac{(6)(2) + (-3)(1) + (2)(-2)}{\sqrt{6^2 + (-3)^2 + 2^2}\sqrt{2^2 + 1^2 + (-2)^2}} =$

$\dfrac{5}{(7)(3)} = \dfrac{5}{21}.$

$\alpha = \cos^{-1}\left(\dfrac{5}{21}\right) \doteq 1.3304 \text{ radians} \doteq 76.2°.$

(#20) For what values of c is the angle between the vectors $\langle 1,2,1\rangle$ and $\langle 1,0,c\rangle$ equal to $60°$?

We use both forms of the dot product!

$$\begin{aligned}
\langle 1,2,1\rangle \cdot \langle 1,0,c\rangle &= |\langle 1,2,1\rangle| * |\langle 1,0,c\rangle| * \cos(60°) \\
1+0+c &= \sqrt{1+4+1}\sqrt{1+0+c^2}\left(\frac{1}{2}\right) \\
1+c &= \frac{1}{2}\sqrt{6}\sqrt{1+c^2} \\
2(1+c) &= \sqrt{6}\sqrt{1+c^2} \quad \text{Square both sides.} \\
4\left(1+2c+c^2\right) &= 6\left(1+c^2\right) \\
4+8c+4c^2 &= 6+6c^2 \\
2c^2-8c+2 &= 0 \\
c^2-4c+1 &= 0 \Rightarrow c = 2\pm\sqrt{3}.
\end{aligned}$$

(#24) The vector projection of b onto a is $\mathbf{Proj_a b} = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

$$\left(\frac{\langle 1,6,-2\rangle \cdot \langle 2,-3,1\rangle}{\langle 2,-3,1\rangle \cdot \langle 2,-3,1\rangle}\right)\langle 2,-3,1\rangle = \left(\frac{2-18-2}{4+9+1}\right)\langle 2,-3,1\rangle = -\frac{9}{7}\langle 2,-3,1\rangle.$$

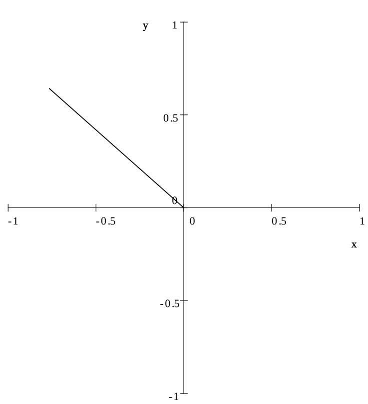
So we see that the projection vector points in the *opposite* direction of **a**.

The scalar projection is the length of $\mathbf{Proj_a b}$ AND we also use the direction. If it points in the opposite direction of **a**, then we take the negative of the magnitude! Thus, the value of the scalar projection is

$$\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{-18}{\sqrt{14}} = -\frac{9\sqrt{14}}{7}.$$

(#30) Tricky wording!

If the direction is N50°W, then start pointing north and *then* go 50° west, which is counterclockwise.



So the vector makes a 50° with respect to the positive y-axis!

In standard position, however, we have $\theta = 140°$.

Thus, the unit direction vector must be $\langle\cos(140°), \sin(140°)\rangle$.

We can always rewrite a vector in this form:

$$\mathbf{v} = |\mathbf{v}| * (\text{unit direction vector associated with } \mathbf{v})\,.$$

Thus, we have

$$\mathbf{F} = 20\left\langle \cos\left(140^\circ\right), \sin\left(140^\circ\right)\right\rangle = \left\langle 20\cos\left(140^\circ\right), 20\sin\left(140^\circ\right)\right\rangle$$

Thus, the horizontal component is $20\cos\left(140^\circ\right)\mathbf{i}$ and the vertical component is $20\sin\left(140^\circ\right)\mathbf{j}$.
The displacement vector is 4 feet due west. That's $\overrightarrow{PQ} = \langle -4, 0\rangle$.

$$\begin{aligned} Work &= \mathbf{F}\cdot\overrightarrow{PQ} = \left\langle 20\cos\left(140^\circ\right), 20\sin\left(140^\circ\right)\right\rangle \cdot \langle -4, 0\rangle \\ &= -80\cos\left(140^\circ\right) \doteq 61.3 \quad \text{work units.} \end{aligned}$$

Since the west displacement vector and the force vector have an angle of separation $\alpha = 40^\circ$, we can use the geometric definition.

$$\begin{aligned} Work &= \mathbf{F}\cdot\overrightarrow{PQ} = |\mathbf{F}| * \left|\overrightarrow{PQ}\right| * \cos\left(40^\circ\right) \\ &= (20)\,(4)\cos\left(40^\circ\right) \\ &= 80\cos\left(40^\circ\right) \doteq 61.3 \quad \text{work units.} \end{aligned}$$

(#36) As we saw in class, we use the vector $\mathbf{u} = \langle 1, 1, 1\rangle$ to represent the diagonal of the cube.
We will use the diagonal across the face which lies in the xy-plane: $\mathbf{v} = \langle 1, 1, 0\rangle$.
We want the angle between these two vectors (anchored at the origin).

$$\cos(\alpha) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}| * |\mathbf{v}|} = \frac{\langle 1,1,1\rangle \cdot \langle 1,1,0\rangle}{|\langle 1,1,1\rangle| * |\langle 1,1,0\rangle|} = \frac{1+1+0}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{6}}.$$

We have $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \doteq 0.6155 \doteq 35.3^\circ$.