Gravitational & Electric Fields: Same Thing, Really!
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1. In a good General Physics I & II sequence, the lecturer should expect that students can derive certain formulas through integration. In particular, gravitational forces are vector sums, and, hence, if the student can integrate vector quantities over various domains, then the student can theoretically calculate gravitational force fields generated by any configuration of mass. Also, if we substitute a (positive) charge distribution for mass, then we can similarly calculate the electric field vectors.

2. A Quick Review...

   Gravitational forces are attractive. Let's assume that we want to evaluate the gravitational field at some point in space. Typically, it is easier to define the mass distribution (preferably in some convenient manner) with respect to the origin, and then calculate gravitational field vectors away from the origin.

   If a point mass M is located at the origin, and our mass m is located at the point (x, y, z), then M's force of attraction on our m is

$$|\mathbf{F}| = \frac{GMm}{r^2},$$

   where r is the distance between the two masses. We note that this is the "Inverse Square Law" where G is some constant, and we have

$$r = \sqrt{x^2 + y^2 + z^2}.$$

   Thus, we have the magnitude of the force of attraction and we know that the force vector must point back toward the origin.

   We recall that we had the special vector field

$$\mathbf{x} = \langle x, y, z \rangle ,$$

   the radial position vector field. At point (x, y, z), the field vector was exactly equal to the vector generated from the origin to the point (x, y, z). Thus, the field is always radial and outward, and we generate longer vectors when we are farther away from the origin.

   We note that

$$|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2} = r \quad \text{and} \quad |\mathbf{F}| = \frac{GMm}{|\mathbf{x}|^2}.$$

   So in order to generate the vector field $\mathbf{F}$, we need the unit vector which points *toward* the origin. That would be

$$-\frac{\mathbf{x}}{|\mathbf{x}|} = \left\langle \frac{-x}{\sqrt{x^2 + y^2 + z^2}}, \ \frac{-y}{\sqrt{x^2 + y^2 + z^2}}, \ \frac{-z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle .$$

Thus, the gravitation field vector at (x, y, z) must be the unit direction vector multiplied by the magnitude and we have

$$\begin{aligned} \mathbf{F}(x,y,z) &= \frac{GMm}{|\mathbf{x}|^2}\left(-\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \left(-\frac{GMm}{|\mathbf{x}|^3}\right)\mathbf{x} \\ &= -GMm\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle. \end{aligned}$$

The difficult task remains. We will need to adjust this definition for the various mass distributions, and then integrate this function over those distributions to obtain the gravitational field vectors at each point. Scary.

3. So let's do a simple example in 2-D with only two point masses. Thus, we will use the 2-D version:

$$\mathbf{F}(x,y) = -GMm\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle.$$

We will place a 10-unit mass at the origin $A(0,0)$ and a 5-unit mass at the point $B(1,0)$ on the x-axis. We will calculate the field vector at $Q(2,2)$ and we will assume that our mass there is one mass unit $(m = 1)$. The sum of the two attractive forces will be the gravitational field vector.

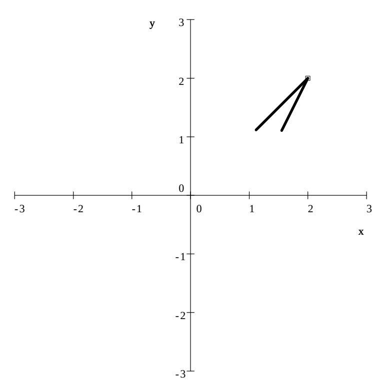
   (a) For the 10-unit mass at the origin, we have

$$\begin{aligned} \mathbf{F}_A(2,2) &= -G(10)(1)\left\langle \frac{2}{(2^2+2^2)^{3/2}}, \frac{2}{(2^2+2^2)^{3/2}} \right\rangle = -G\left\langle \frac{20}{8^{3/2}}, \frac{20}{8^{3/2}} \right\rangle \\ &= -G\left\langle \frac{20}{8\sqrt{8}}, \frac{20}{8\sqrt{8}} \right\rangle = -G\left\langle \frac{5\sqrt{2}}{8}, \frac{5\sqrt{2}}{8} \right\rangle. \end{aligned}$$

   (b) For the 5-unit mass, we need the displacement vector $\overrightarrow{BQ} = \langle 1, 2\rangle$.
   Following the same pattern, we must have

$$\begin{aligned} \mathbf{F}_B(2,2) &= \left(\frac{-GMm}{|displacement\ vector|^2}\right)(unit\ displacement\ vector) \\ &= \left(\frac{-G(5)(1)}{\left(\sqrt{1^2+2^2}\right)^2}\right)\left(\frac{\langle 1,2\rangle}{\sqrt{1^2+2^2}}\right) = -G\left(\frac{\sqrt{5}}{5}\right)\langle 1,2\rangle \end{aligned}$$

   (c) In order to simplify our sketches, we will let $G = 1$.

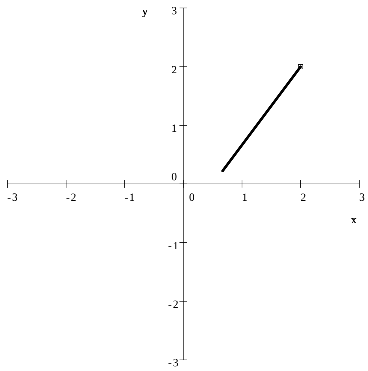


The real universal gravitation constant G is, of course, a different number.

If we anchor both vectors at $(2,2)$, we see that F_A points back toward $A(0,0)$ and F_B points back toward $B(1,0)$.

Our mass at $Q(2,2)$ is simultaneously attracted to the masses at A and B.

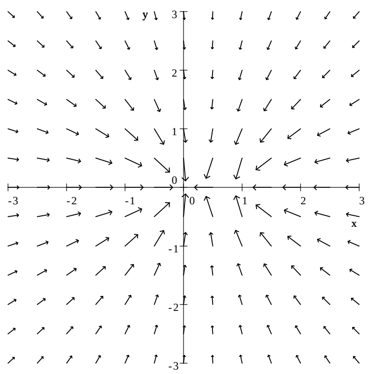
Now we add the two vectors together and anchor the resultant at $(2,2)$. This gives us our field vector $\mathbf{F}(2,2)$ when $M=1$.



So we have

$$\mathbf{F}(2,2) = -G\left\langle \frac{5\sqrt{2}}{8}+\frac{\sqrt{5}}{5},\ \frac{5\sqrt{2}}{8}+\frac{2\sqrt{5}}{5}\right\rangle.$$

This is approximately $-G\langle 1.33, 1.78\rangle$.



If we plot the field at other points, we get a better sense of how the two masses generate the field.

We note that my software automatically scales the field vectors so that they do not run over each other.

We see that the field is strongest (longest vectors) near the points $(0,0)$ and $(1,0)$, and that the field becomes decidedly weaker as we move farther away.

(d) Now comes the cool part. We calculate electric fields in the same manner, except that charges with the same polarity (both positive, for example) generate *repulsive* forces.

If we rephase the gravity problem (as above), then we place a $(+10)$ charge at the origin $A(0,0)$ and a $(+5)$ charge at the point $B(1,0)$ on the x-axis. We will calculate the field vector at $Q(2,2)$ and when we calculate electric field, we always place a $(+1)$ "test charge" at our point interest.

We will let our universal electric field constant be k.

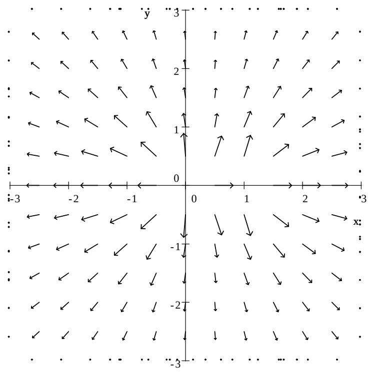
All of the previous calculations are identical, since electric field strength also follows the Inverse Square Law.

Thus, at $(2,2)$, we have (we will call our electric field "$\mathbf{E}$")

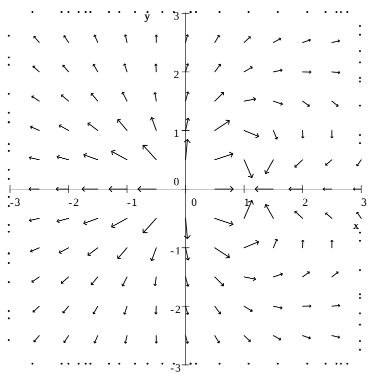
$$\mathbf{E}(2,2) = k\left\langle \frac{5\sqrt{2}}{8}+\frac{\sqrt{5}}{5},\ \frac{5\sqrt{2}}{8}+\frac{2\sqrt{5}}{5}\right\rangle$$

and, in general, the electric field vectors will point in the opposite direction from the previously calculated gravitation vectors.

[See next page for the characterization of $\mathbf{E}$.]



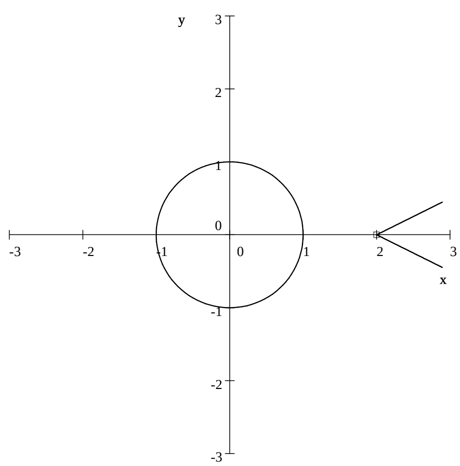
(e) Notice what happens if we change the $(+5)$ charge at $B\,(1,0)$ to a (-5) charge. The charge at $A\,(0,0)$ will be repulsive and the charge at $B\,(1,0)$ will be attractive.



4. Here's another 2-D problem. Suppose we have a ring of positive charge on the unit circle.

   The charge density on the ring is σ units per unit length.

   We want to calculate the electric field vector at $P\,(2,0)$.



I've sketched in some sample vectors at $P\,(2,0)$ associated with pieces of charge on the ring.

The vector which points slightly upward would be the force associated with the charge at $(0,-1)$ on the unit circle. Similarly, the vector pointing slightly downward is associated with the charge at $(0,1)$.

We note, by symmetry, the vertical components of these two vectors would cancel out!

Okay! That's our theme for the remainder of this article. We want to look at symmetric charge distributions and we will calculate electric field (or gravitational field) at points which allow us to cancel out some of the force components *by symmetry.*

(a) Let's set up this problem.

We will parameterize the circle using our standard:

$$\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \end{aligned}$$

for $0 \le t \le 2\pi$. $[(\cos(t), \sin(t)) = \text{our initial point}]$

Since t also represents arc length (one circumference), we have the charge differential

$$dq = \sigma * dt.$$

The displacement vector from $(\cos(t), \sin(t))$ to $(2, 0)$ is

$$\langle 2 - \cos(t), -\sin(t) \rangle.$$

Thus, for $0 \le t \le 2\pi$, we can generate a force vector:

$$\begin{aligned} \mathbf{F} &= \left(\frac{k(\sigma\ dt)(+1)}{|displacement\ vector|^2} \right)(unit\ displacement\ vector) \\ &= \left(\frac{k\sigma\ dt}{|displacement\ vector|^2} \right)\left(\frac{displacement\ vector}{|displacement\ vector|} \right) \\ &= \left(\frac{k\sigma\ dt}{|displacement\ vector|^3} \right)(displacement\ vector). \end{aligned}$$

Our shorthand formula allows us to calculate the displacement vector, $\langle f, g \rangle$, and this gives us

$$\mathbf{F} = k\sigma \left\langle \frac{f}{(f^2+g^2)^{3/2}}, \frac{g}{(f^2+g^2)^{3/2}} \right\rangle dt.$$

This allows us to set up the integrations extremely quickly! You should memorize this form for Physics II!

In our example, we have

$$\begin{aligned} f &= 2 - \cos(t) \\ g &= -\sin(t). \end{aligned}$$

We note that if we integrate the vertical components around the unit circle, the resulting integral must be zero by symmetry! So we need only integrate the horizontal component. We must integrate the force vectors from 0 to 2π. The horizontal component of $\mathbf{E}(2, 0)$ is equal to

$$k\sigma \int_0^{2\pi} \frac{2 - \cos(t)}{\left((2-\cos(t))^2 + (-\sin(t))^2\right)^{3/2}}\, dt.$$

Unfortunately, this does *not* have a closed form antiderivative, so we must numerically approximate it.

The resulting value is approximately $1.9566k\sigma$.

Since the total charge on the ring is $Q = 2\pi\sigma$, we have $\sigma = \frac{Q}{2\pi}$, and the electric field vector is approximately

$$\left\langle 1.9566k\left(\frac{Q}{2\pi}\right),\ 0 \right\rangle = \langle 0.3114kQ,\ 0 \rangle.$$

(b) Notice that the following "nice thought" does NOT work...

Could we represent all of the charge on the ring by an equivalent point charge at the center of the ring (the origin)?

NO.

The total charge is Q (as stated above) and our test charge is $(+1)$ at the point $P\,(2, 0)$. The displacement vector is $\langle 2, 0\rangle$.

$$\mathbf{E}\,(2,0) = \frac{kQ\,(+1)}{(2^2+0^2)^{3/2}}\,\langle 2, 0\rangle = kQ\left\langle \frac{2}{4^{3/2}},\ 0\right\rangle = \langle 0.25kQ,\ 0\rangle\,.$$

The two results do NOT match. Because of the Inverse Square Law, we cannot use "center of mass" techniques to simplify problems like this! It's not great news, but since we have software which can approximate ugly integrals, most problems (up through 3-D) are generally tractable.

5. Here's a classic set-up. You can probably find the answer to this one in any Physics II text.

Suppose we have an infinite line of charge on the y-axis. The density is σ units per unit length.

We want to calculate the electric field at $(1, 0)$.

The obvious parameterization is

$$\begin{aligned} x &= 0 \\ y &= t \end{aligned}$$

for $-\infty < t < +\infty$.

We have

$$dq = \sigma * dt$$

and the displacement vector for the charge at $(0, t)$ to our point $(1, 0)$ is

$$\langle 1,\ -t\rangle\,.$$

Again, the vertical components will all cancel out by symmetry, so you need only integrate the horizontal components.

Show that

$$\mathbf{E}\,(1,0) = \langle 2k\sigma,\ 0\rangle\,.$$

Also, what happens when we move the $(+1)$ test charge to $P\,(a, 0)$?

$$\mathbf{E}\,(a,0) = \langle ???,\ 0\rangle\,.$$

6. Now let's think about electric field in 3-D. Instead, our force vectors look like this, if our displacement vector is $\langle f, g, h \rangle$:

$$\mathbf{F} = k\sigma \left\langle \frac{f}{(f^2+g^2+h^2)^{3/2}}, \frac{g}{(f^2+g^2+h^2)^{3/2}}, \frac{h}{(f^2+g^2+h^2)^{3/2}} \right\rangle dt.$$

For our 3-D questions, all of charge distributions will be symmetric with respect to the z-axis, so integrating the first two components should give us zero by symmetry.

Let's try our ring of charge again. The unit circle lies in the xy-plane, $x^2 + y^2 = 1$.

The density is σ units per unit length again.

$$\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \\ z &= 0 \end{aligned}$$

for $0 \leq t \leq 2\pi$.

We will calculate the electric field at $(0,0,1)$. The initial point is $(\cos(t), \sin(t), 0)$.

The displacement vector is $\langle -\cos(t), -\sin(t), 1 \rangle$. The third (z) component is

$$k\sigma \left(\frac{1}{\left((-\cos(t))^2 + (-\sin(t))^2 + 1^2\right)^{3/2}} \right) dt = k\sigma \left(\frac{1}{2^{3/2}} \right) dt = k\sigma \left(\frac{\sqrt{2}}{4} \right) dt.$$

The magnitude of the electric field vector must be

$$\int_0^{2\pi} k\sigma \left(\frac{\sqrt{2}}{4} \right) dt = k\sigma \left(\frac{\pi\sqrt{2}}{2} \right).$$

Since the total charge of the ring is $Q = 2\pi\sigma \Rightarrow \sigma = \dfrac{Q}{2\pi}$, we have

$$k \left(\frac{Q}{2\pi} \right) \left(\frac{\pi\sqrt{2}}{2} \right) = kQ \left(\frac{\sqrt{2}}{4} \right).$$

The electric field vector is

$$\mathbf{E}(0,0,1) = \left\langle 0,\ 0,\ kQ \left(\frac{\sqrt{2}}{4} \right) \right\rangle.$$

Make the appropriate adjustments and find the electric field at $P(0,0,a)$!

7. Suppose the charge is distributed over a circular disk in the xy-plane, $x^2 + y^2 \leq 1$.

   The charge density is now σ units per *square unit.*

   You will calculate the electric field at $(0, 0, 1)$.

   Every piece of charge in the disk depends on *two* variables, r and θ. Thus, the initial point is $(r\cos(\theta),\ r\sin(\theta),\ 0)$ and our displacement vector (in polar) is

$$\langle -r\cos(\theta),\ -r\sin(\theta),\ 1\rangle$$

   for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

   Set up and evaluate the appropriate *double integral* of the z-axis component. We already know that the first two components will cancel by symmetry. Don't forget the Jacobian!!!

   Also, use the fact that the total charge on the disk is

$$Q = (Area\ of\ R)\,\sigma = \pi\sigma \Rightarrow \sigma = \frac{Q}{\pi}$$

   and express your final answer in terms of Q (rather than σ).

   Now make the appropriate adjustments and find the electric field at $P(0, 0, a)$.

8. Now suppose that the entire xy-plane is charged! The charge density is still σ units per square unit.

   Adjust your integration from (#7) and let $r:\ \ 0 \to +\infty$.

   First, calculate the electric field at $P(0, 0, 1)$, and then adjust the integration to find the electric field at $P(0, 0, a)$.

   You can check your answer against any Physics II text.

9. This situation typically does not pop up in Physics II because of Gauss' Law, so I will phrase it as a gravitation problem.

   Suppose we have the solid unit sphere $\rho \leq 1$ with uniform density σ mass units per cubic unit.

   Find the gravitational field vector at $P(0, 0, 2)$ if the point mass there is $m = 1$.

   Remember that you only need to integrate the z-component of the force vectors over each $dm = \sigma\ dV$ differential in the solid sphere. Use Spherical Coordinates, of course, in the triple integral. Also, note that since gravitational forces are attractive, your final answer should be pointing downward (in the negative z-direction).

   Hint: Since there is no θ in the integrand, you can actually just approximate the $d\rho\, d\phi$ portion (in a double integral) and then multiply that answer by 2π.

   Since the mass of the unit sphere is $M = (Volume)\,\sigma = \frac{4}{3}\pi\sigma$, we can substitute $\sigma = \frac{3M}{4\pi}$ into our final answer and determine the field vector in terms of the attracting mass.