Solutions to Final Exam for Analytic Geometry and Calculus III – MATH 2421
Fall 2004

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(#1) Clues.

(i) When $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, the value of $\mathbf{u} \cdot \mathbf{v}$ is **(a)**. The angle of separation is $\alpha = \frac{\pi}{2}$, so the dot product must be zero.

(ii) If $\mathbf{F}(x, y, z)$ is a conservative vector field, then the value of $\nabla \cdot \mathbf{F}$ is **(c)**. The CURL is the zero vector. Conservative fields are irrotational. The divergence could be just about anything.

(iii) If $\mathbf{u} = \mathbf{v} \times \mathbf{w}$, then the value of $\mathbf{Proj_v u}$ is **(b)**. We have that $\frac{(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$ must equal zero. So the vector projection must be the zero vector.

(#2) An object is moved from $P(1, 3, -5)$ to $Q(4, 4, -4)$ along a straight line segment.

(a) Give the parameterization of the path (from P to Q) using $0 \le t \le 1$.
$\mathbf{r}(t) = \langle 3t + 1,\ 1t + 3,\ 1t - 5 \rangle$.

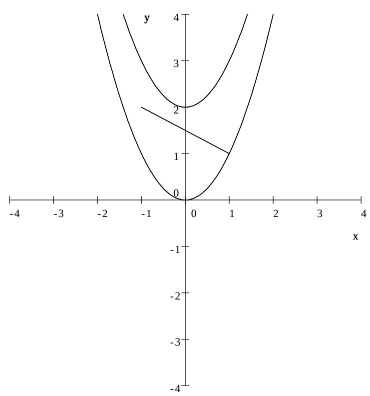
(b) If the force vector is constant, $\mathbf{F} = \langle 3, 6, 5 \rangle$, then how much work does $\mathbf{F}$ accomplish moving the object from P to Q?
$\langle 3, 6, 5 \rangle \cdot \langle 3, 1, 1 \rangle = 9 + 6 + 5 = 20$ work units.

(c) Suppose $\mathbf{F}$ is the force field $\mathbf{F}(x, y, z) = \left\langle x^2,\ -y,\ z \right\rangle$.
$d\mathbf{r} = \langle 3, 1, 1 \rangle\ dt$
Work $= \int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle (3t+1)^2,\ -(t+3),\ t-5 \right\rangle \cdot \langle 3, 1, 1 \rangle\ dt$
In case anyone cares, the actual answer is

$$\int_0^1 \left(3(3t+1)^2 - (t+3) + t - 5\right) dt = 13 \text{ work units.}$$

(#3) Suppose we have a 2-variable function $f(x, y) = y - x^2 = k \Rightarrow y = x^2 + k$.

(a) On the axes below, sketch in the level curves $f(x, y) = k$ for $k = 0$ and $k = 2$.



The upper parabola corresponds to $k = 2$.

As the parabolas move down the y-axis, the values of k DECREASE. This is a significant observation for later...

(b) It should be clear to you that the point $(1,1)$ must lie on the level curve associated with $k=0$, since $f(1,1)=0$. Sketch in $\nabla f(1,1)$ at the point $(1,1)$.
$\nabla f = \langle -2x, 1\rangle \Rightarrow \nabla f(1,1) = \langle -2, 1\rangle$.
The gradient vector should be normal to the parabola (level curve) and, in this case, its length should be $\sqrt{5}$.

(c) Suppose we wanted to find the relative extrema for f.
Show that f has NO critical points.
$f_x = 2x = 0 \Rightarrow x = 0$.
$f_y = 1 \neq 0$. Thus, f has no critical points.

(d) Now suppose that our domain of interest (D) is the area $0 \le y \le -x^2 + 4x$.
$x = t$
$y = -t^2 + 4t$, for $0 \le t \le 4$.
$f\left(t,\ -t^2+4t\right) = -t^2 + 4t - t^2 = -2t^2 + 4t$.
$f' = -4t + 4 = 0 \Rightarrow t = 1$.
$f'' = -4$ so it is relative maximum along this boundary.

(e) It was a relative maximum along that boundary.
Since f is differentiable everywhere and no relative extrema was found in D, then it must be true that the absolute extrema must be found on the boundaries of D. In this case, the point $(1,3)$ must be the absolute MAXIMUM of f in D.

(#4) Suppose f is a function of u, v, and w.

Suppose $u(r,\theta) = r\theta$ and $v(r) = e^r$ and $w(r,\theta) = \dfrac{\theta}{r}$.

Which expression below is equivalent to $\dfrac{\partial f}{\partial \theta}$?

The basic Chain Rule is $\dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial \theta} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial \theta}$,

but v doesn't depend on θ, so the correct answer is **(c).**

(#5) We already know that the function $f(x,y) = 3x^2 + 3xy + y^2$ has a critical point at $(0,0)$.
What is the classification of this critical point? We already know that $f_x = 0$ and $f_y = 0$ at $(0,0)$.

At $(0,0)$, we have $f_{xx} = 6$, $f_{yy} = 2$, and $f_{xy} = 3$.

$d = f_{xx}f_{yy} - (f_{xy})^2 = 12 - 9 = 3 > 0$. Since both f_{xx} and f_{yy} are also positive, we have a relative minimum.

The correct answer is **(a)**.

(#6) If $\mathbf{F} = \left\langle x - z,\ xz^2,\ \dfrac{y}{z}\right\rangle$, what is the value of $\nabla \times \mathbf{F}$?

$$P = x - z, \quad Q = xz^2, \quad R = \frac{y}{z}.$$

$$\nabla \times \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$$

$$= \left\langle \frac{1}{z} - 2zx,\ -1 - 0,\ z^2 - 0\right\rangle.$$

The correct answer is **(b)**.

(#7) If $(x,y,z) = \left(-4,\ 4,\ -4\sqrt{2}\right)$, then $(\rho,\theta,\phi) =$???

$$\begin{aligned}\rho &= \sqrt{x^2+y^2+z^2} = \sqrt{(-4)^2+4^2+\left(-4\sqrt{2}\right)^2} = \sqrt{16+16+32} = 8.\\ \tan(\theta) &= \frac{y}{x} = \frac{4}{-4} = -1 \text{ and } \theta \text{ is in Quad. II} \Rightarrow \theta = \frac{3\pi}{4}.\\ \cos(\phi) &= \frac{z}{\rho} = \frac{-4\sqrt{2}}{8} = -\frac{\sqrt{2}}{2} = \frac{3\pi}{4}.\end{aligned}$$

The correct answer is **(d)**.

(#8) If the unit tangent vector is $\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}} \right\rangle$, then the unit normal vector $\mathbf{N}(t) =$???

We have

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

and we notice that all we need to know if the sign of the horizontal and vertical components of $\mathbf{N}(t)$.

(It is easy to verify that they are all unit vectors.)

$$\begin{aligned}\mathbf{T}'(t) &= \frac{d}{dt}\left[\left\langle \left(1+t^2\right)^{-1/2}, \frac{t}{\sqrt{1+t^2}} \right\rangle\right]\\ &= \left\langle \left(-\frac{1}{2}\right)\left(1+t^2\right)^{-3/2}(2t), \frac{\sqrt{1+t^2}(1) - t\left(\frac{1}{2} * \frac{2t}{\sqrt{1+t^2}}\right)}{\left(\sqrt{1+t^2}\right)^2} \right\rangle\\ &= \left\langle -\frac{t}{\left(1+t^2\right)^{3/2}}, \frac{\sqrt{1+t^2} - \frac{t^2}{\sqrt{1+t^2}}}{1+t^2} \right\rangle = \left\langle \frac{t}{\left(1+t^2\right)^{3/2}}, \frac{\left(\frac{1+t^2-t^2}{\sqrt{1+t^2}}\right)}{1+t^2} \right\rangle\\ &= \left\langle -\frac{t}{\left(1+t^2\right)^{3/2}}, \frac{1}{\left(1+t^2\right)^{3/2}} \right\rangle = \left(\frac{1}{\left(1+t^2\right)^{3/2}}\right)\langle -t, 1\rangle.\end{aligned}$$

When we factor out the denominators for $\mathbf{N}(t)$, they disapper.

$$\mathbf{N}(t) = \frac{\left(\frac{1}{(1+t^2)^{3/2}}\right)\langle -t,1\rangle}{\left|\frac{1}{(1+t^2)^{3/2}}\right| * |\langle -t,1\rangle|} = \frac{\langle -t,1\rangle}{\sqrt{1+t^2}}.$$

The correct answer is **(c)**.

(#9) Examine the two vector fields below. One of them is a good representation of

$\mathbf{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$. The other one is unreasonable, considering the magnitudes displayed.

Circle the one which best represents **F** and give a one-sentence justification for your choice.

We notice that all vectors in **F** are unit vectors. The configuration is radial outward. It must be the field on the left.

(#10) Suppose we have the surface $x\sin(z) - \cos(yz) = 2 - \frac{\sqrt{3}}{2}$. Find $\frac{\partial z}{\partial x}$ evaluated at $\left(2, \frac{1}{3}, \frac{\pi}{2}\right)$.

We need the implicit differentiation formula. We create the "parent" function by moving everything to the left side.

$$F(x,y,z) = x\sin(z) - \cos(yz) - 2 + \frac{\sqrt{3}}{2}.$$

We must calculate

$$\frac{\partial z}{\partial x} = -\left(\frac{F_x}{F_z}\right).$$

We have

$$F_x = \sin(z), \quad F_z = x\cos(z) + y\sin(yz).$$

This gives us

$$\frac{\partial z}{\partial x} = -\left(\frac{\sin(z)}{x\cos(z) + y\sin(yz)}\right)$$

and substitute in the values for x, y, and z.

$$\left[\frac{\partial z}{\partial x}\right]_{\left(2,\frac{1}{3},\frac{\pi}{2}\right)} = -\left(\frac{\sin\left(\frac{\pi}{2}\right)}{2\cos\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{1}{3}\left(\frac{\pi}{2}\right)\right)}\right) = -\left(\frac{1}{2(0) + \frac{1}{3}\left(\frac{1}{2}\right)}\right) = -6.$$

(#11) Let $\mathbf{F}(x,y) = \left\langle xe^x + 2xy,\ x^2 + y^3 \right\rangle$.

(a) Show that **F** is conservative. If $Q_x = P_y$, then **F** is conservative.
$Q_x = 2x$
$P_y = 2x.$✓
$f(x,y) =$

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for any simple path which begins at $P(0,0)$ and ends at $Q(\ln(2), 1)$.

The Fundamental Theorem for Line Integrals applies. Find the potential function by integrations.

$$\int P dx = \int (xe^x + 2xy)\, dx = xe^x - e^x + x^2 y + \alpha(y).$$

$$\int Q dy = \int \left(x^2 + y^3\right) dy = x^2 y + \frac{y^4}{4} + \beta(x).$$

Any terms with x and y must match in both integrations.✓
Now collect one of each other term (no duplications).

$$f(x,y) = x^2 y + xe^x - e^x + \frac{y^4}{4}.$$

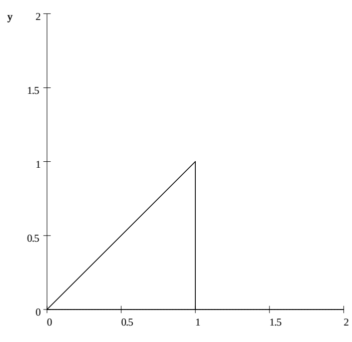
The Fundamental Theorem says that the amount of work performed by $\mathbf{F}$ is path independent.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= f(\ln(2),\ 1) - f(0,0) \\
&= (\ln(2))^2(1) + (\ln(2))e^{\ln(2)} - e^{\ln(2)} + \frac{1^4}{4} - (0+0-1+0) \\
&= (\ln(2))^2 + 2\ln(2) - 2 + \frac{1}{4} + 1 = (\ln(2))^2 + 2\ln(2) - \frac{3}{4}.
\end{aligned}$$

(#12) As is stands, the following double integral is impossible to evaluate...

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = ???$$

(a) First, sketch the region of integration R.



dx dy
left $\to$ *right*
$x:\ \ y \to 1$
$y:\ \ 0 \to 1$
This becomes:
dy dx
lower $\to$ *upper*
$y:\ \ 0 \to x$
$x:\ \ 0 \to 1.$

(b) Now use Fubini to evaluate the double integral.

$$\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1).$$

Inner:

$$e^{x^2}\int_0^x dy = xe^{x^2}.$$

Outer:

$$\int_0^1 xe^{x^2}\,dx \Rightarrow u = x^2,\ du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du.$$

This gives us

$$\frac{1}{2}\int e^u\,du \Rightarrow \frac{1}{2}e^u \Rightarrow \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).\checkmark$$

(#13) If $f(x,y,z) = xy + xz + yz$, find the value of the total differential df if x moves from 2 to 2.2, y moves from 3 to 2.9, and z moves from 4 to 4.01.

$$df = f_x dx + f_y dy + f_z dz = (y+z)\,dx + (x+z)\,dz + (x+y)\,dz$$

We have $dx = +0.2,\ dy = -0.1$, and $dz = +0.01$.

We substitute in $x = 2,\ y = 3$, and $z = 4$.

$$df = (3+4)(0.2) + (2+4)(-0.1) + (2+3)(0.01) = +0.85.$$

We expect f to increase by about 0.85 units.

(#14) Find the (rectangular coordinates) equation of the plane which is perpendicular to the yz-plane and contains the origin and the point $(0, 1, 2)$.

If the plane is perpendicular to the yz-plane, then the equation only contains y and z.

Thus, the normal vector is $\mathbf{n} = \langle 0, b, c \rangle$. If it contains the origin, then the standard form must be

$$\begin{aligned} 0(x-0) + b(y-0) + c(z-0) &= 0 \\ by + cz &= 0 \end{aligned}$$

If it contains $(0, 1, 2)$, then we must have

$$\begin{aligned} b(1) + c(2) &= 0 \\ b &= -2c. \end{aligned}$$

Any pair satisfying this condition will give us the correct plane, so we choose $c = 1$ and $b = -2$.

$$-2y + z = 0 \quad \text{or} \quad z = 2y.$$

As an alternative, we could use the normal definitions for a plane.

The normal vector to our plane must be perpendicular to the normal of the yz-plane, and it must be perpendicular to vector which connects $(0, 0, 0)$ and $(0, 1, 2)$. Sounds like the cross product.

The normal to the yz-plane is $\langle 1, 0, 0 \rangle$ since the equation of the plane is $x = 0$.

$$\langle 1, 0, 0 \rangle \times \langle 0, 1, 2 \rangle = \langle 0, -2, 1 \rangle.$$

Thus, we obtain

$$\begin{aligned} 0(x-0) - 2(y-0) + 1(z-0) &= 0 \\ z &= 2y. \checkmark \end{aligned}$$

(#15) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = \left\langle \sin\left(x^3\right), \frac{x^3}{3} + xy^2 + \ln(1+y) \right\rangle$ and C is the closed path which bounds the semicircle $x^2 + y^2 \leq 1$ with $y \geq 0$.

Sounds like a job for Green's Theorem! The circulation density is

$$Q_x - P_y = x^2 + y^2 - 0 = x^2 + y^2 = r^2.$$

Our region is clearly polar.

$r\ dr\ d\theta$

$r: \quad 0 \to 1$

$\theta: \quad 0 \to \pi \quad [y \geq 0]$.

$$\int_0^\pi \int_0^1 r^2\ r\ dr\ d\theta = \frac{\pi}{4}.$$

(#16) The region of integration R is the portion of the circle in Quadrant I whose center is located at $(0,1)$ with radius 1.

If $z = g(x,y) = \sqrt{x^2+y^2}$ (the upper cone), and the density function is $\sigma(x,y,z) = x = r\cos(\theta)$. Find the mass for the portion of the cone which is above R.

I hope you remember that the polar equation of the curve is $r = 2\sin(\theta)$, $0 \le \theta \le \pi/2$.

We need the surface area differential.

$$\begin{aligned} dS &= \sqrt{1+(g_x)^2+(g_y)^2}dA \\ &= \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}dA \\ &= \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}dA = \sqrt{2}\ dA. \end{aligned}$$

We insert the density function in the integrand also.

$$\begin{aligned} m &= \iint_R x\sqrt{2}dA \\ &= \sqrt{2}\int_0^{\pi/2}\int_0^{2\sin(\theta)} r\cos(\theta)\ r\ dr\ d\theta \\ &= \sqrt{2}\int_0^{\pi/2}\int_0^{2\sin(\theta)} r^2\cos(\theta)\ dr\ d\theta \end{aligned}$$

Inner:

$$\sqrt{2}*\cos(\theta)\int_0^{2\sin(\theta)} r^2 dr = \frac{8\sqrt{2}}{3}\sin^3(\theta)\cos(\theta).$$

Outer:

$$\begin{aligned} \frac{8\sqrt{2}}{3}\int_0^{\pi/2}\sin^3(\theta)\cos(\theta)\,d\theta &= \frac{8\sqrt{2}}{3}\left[\frac{\sin^4(\theta)}{4}\right]_0^{\pi/2} \\ &= \frac{8\sqrt{2}}{3}\left(\frac{1}{4}\right) = \frac{2}{3}\sqrt{2}.\checkmark \end{aligned}$$

The Outer integral is a u-substitution, $u = \sin(\theta)$, $du = \cos(\theta)\,d\theta$.

$$\int u^3 du \Rightarrow \frac{u^4}{4} \Rightarrow \left[\frac{\sin^4(\theta)}{4}\right]_0^{\pi/2} = \frac{1}{4}.$$

(#17) Let S be the portion of the hyperbolic paraboloid $z = x^2 - y^2$ inside the right circular cylinder $x^2 + y^2 = 1$.

The field is

$$\mathbf{F}(x, y, z) = \left\langle \frac{1}{2}x,\ 3y,\ z \right\rangle.$$

We want

$$\int_S (\mathbf{F} \cdot \mathbf{n})\, dS$$

where n is the *upward* normal with respect to S.

We have $P = \frac{1}{2}x,\ Q = 3y$, and $R = z$.

Our surface is $z = g(x, y) = x^2 - y^2$.

$$g_x = 2x \quad \text{and} \quad g_y = -2y.$$

We are evaluating $(\mathbf{F} \cdot \mathbf{n})$ over S. We smash S down onto the xy-plane and obtain the circle R.

$r\ dr\ d\theta$

$r:\quad 0 \to 1$

$\theta:\quad 0 \to 2\pi.$

$$\int_S (\mathbf{F} \cdot \mathbf{n})\, dS = \int_R (-Pg_x - Qg_y + R)\, dA$$

First, we substitute in rectangular, remembering to also substitute $z = x^2 - y^2$.

$$\int_R \left(-\left(\frac{1}{2}x\right)(2x) - (3y)(-2y) + \left(x^2 - y^2\right) \right) dA = 5\int_R y^2 dA.$$

Now change to polar.

$$5\int_0^{2\pi}\int_0^1 r^2 \sin^2(\theta)\ r\ dr\ d\theta = \frac{5}{4}\pi.$$

Inner:

$$\int_0^1 r^3 dr = \frac{1}{4}.$$

Outer:

$$5\int_0^{2\pi} \sin^2(\theta)\, d\theta = 5\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = 5\pi.$$

The product is $\frac{5}{4}\pi.$✓

(#18) Let Q be the solid hemisphere $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$, and let S be its surface.

Let $\mathbf{F}(x, y, z) = \langle xze^z,\ -yze^z,\ z^2 \rangle$ and $\mathbf{n} =$ the outward normal with respect to S.

Evaluate the outward flux integral $\oint_S (\mathbf{F} \cdot \mathbf{n})\, dS$. We use the Divergence Theorem.

$$\nabla \cdot \mathbf{F} = ze^z - ze^z + 2z = 2z.$$

We integrate the flux density over the interior of S.

$\rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

$\rho:\quad 0 \to 1$

$\phi:\quad 0 \to \pi/2\quad$ [since $z \geq 0$]

$\theta:\quad 0 \to 2\pi.$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 2\rho \cos(\phi)\, \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta = \frac{\pi}{2}.$$

We note that $2z = 2\rho \cos(\phi)$.

The triple integral factors neatly into the three integrals.

$$\int_0^1 2\rho^3 d\rho = \frac{1}{2}.$$

$$\int_0^{\pi/2} \sin(\phi)\cos(\phi)\ d\phi = \frac{1}{2}.$$

We let $u = \sin(\phi)$, $du = \cos(\phi)\, d\phi$, etc.

The integral in θ is equal to 2π. The product is

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2\pi) = \frac{\pi}{2}.\checkmark$$