Solutions to Test on Ch. 9 & 10 – MATH 2421
Spring 2004

(#1) In cylindrical coordinates, the *surface* $r = \cos(\theta)$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, assumes that z can be anything. How should we name this surface?

We recall that $r = \cos(\theta)$ is a *circle* of radius $R = \dfrac{1}{2}$ centered at $\left(\dfrac{1}{2}, 0\right)$ in the xy-plane. If we duplicate this circle in the z-axis direction, we have a right circular cylinder.

The correct answer is **(c)**.

(#2) With which angle θ do we associate the plane $y = -x$?

The slope of this line in the xy-plane is $m = -1 = \tan(\theta)$.

The correct answer is **(d)** $\theta = \dfrac{3\pi}{4}$

(#3) If $\mathbf{u} = \langle -2, 0, -4 \rangle$ and $\mathbf{v} = \langle 1, -1, 1 \rangle$, then which vector is equal to $\mathbf{Proj_v u}$?

$\mathbf{Proj_v u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\dfrac{\langle -2,0,-4\rangle \cdot \langle 1,-1,1\rangle}{\langle 1,-1,1\rangle \cdot \langle 1,-1,1\rangle}\right)\langle 1,-1,1\rangle = \left(\dfrac{-2+0-4}{1+1+1}\right)\langle 1,-1,1\rangle =$

$\left(\dfrac{-6}{3}\right)\langle 1,-1,1\rangle = \langle -2,\ 2,\ -2\rangle$. The correct answer is **(c)**.

(#4) A cable runs from $P(1,0)$ to $Q(6,12)$. The tension vector has the same direction as $\overrightarrow{PQ}$, but has magnitude 100 pounds. Which vector below represents the tension vector?

$\overrightarrow{PQ} = \langle 5,\ 12\rangle$. The unit vector associated with this vector is

$$\frac{1}{|\langle 5,\ 12\rangle|}\langle 5,\ 12\rangle = \frac{1}{13}\langle 5,\ 12\rangle = \left\langle \frac{5}{13},\ \frac{12}{13}\right\rangle.$$

Now multiply by this 100. The resulting vector will have the correct length and direction!

The correct answer is **(a)** $\dfrac{100}{13}\langle 5, 12\rangle$.

(#5) $x^2 + y^2 + z^2 + 2x - 2y = 25$. What is the radius of this sphere? Complete the square.

$\left(x^2 + 2x + 1^2\right) + \left(y^2 - 2y + (-1)^2\right) + z^2 = 25 + 1^2 + (-1)^2$

$(x+1)^2 + (y-1)^2 + (z-0)^2 = 27.$

$R = \sqrt{27} = 3\sqrt{3}$. [The center is located at $C(-1,\ 1,\ 0)$.]

The correct answer is **(d)**.

(#6) These are right circular cones and the central axis is the y-axis. It must be $y^2 = x^2 + z^2$.

The correct answer is **(e)**.

(#7) Suppose we want to move an object from $P(1,-3,0)$ to $Q(15,5,-6)$.

(a) $\overrightarrow{PQ} = \langle 14,\ 8,\ -6\rangle$

(b) Write down the parametric equations which define the line in 3-D which passes through P and Q.
$x = 14t + 1$
$y = 8t - 3$
$z = -6t + 0$

(c) Suppose that the force $\mathbf{F} = \langle 2, 1, -3\rangle$ moves the object from P to Q. Find the number of work units accomplished by the force.
Work $= \mathbf{F}\cdot\overrightarrow{PQ} = \langle 2,1,-3\rangle \cdot \langle 14,\ 8,\ -6\rangle = 28 + 8 + 18 = 54$ work units.

(#8) We have two planes: $x + 2y + 3z = 6$ and $-3x + 2y + 6z = 12$.

Find the acute $\left(0 \le \alpha < \frac{\pi}{2}\right)$ angle between the planes, also called the dihedral angle.

The answer is *not* one of the standard angles. We use the dot product between the two normal vectors.

$\mathbf{n}_1 = \langle 1,2,3\rangle \qquad \mathbf{n}_2 = \langle -3,2,6\rangle$

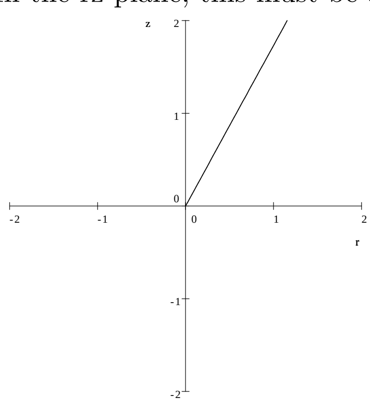
$$\cos(\alpha) = \frac{\langle 1,2,3\rangle \cdot \langle -3,2,6\rangle}{|\langle 1,2,3\rangle| * |\langle -3,2,6\rangle|} = \frac{-3+4+18}{\sqrt{1^2+2^2+3^2}\sqrt{(-3)^2+2^2+6^2}} = \frac{19}{7\sqrt{14}}.$$

$$\alpha = \cos^{-1}\left(\frac{19}{7\sqrt{14}}\right) \doteq 43.5^\circ.$$

(#9) The upper cone $z = \sqrt{3}\sqrt{x^2+y^2}$ can easily be transformed into cylindrical and spherical coordinates.

(a) Make one substitution and find the equation in cylindrical coordinates.
$z = \sqrt{3} * r$.

(b) Based on your previous work, show me what the trace (slice) looks like if I choose any of the θ-planes (rz-planes) through the z-axis.
In the rz-plane, this must be a straight line.



(c) Tougher. By making a few more substitutions and then some cancelling, show that the equation in spherical coordinates is $\phi =$ *a constant angle*.
$z = \sqrt{3} * r \Rightarrow \rho\cos(\phi) = \sqrt{3}\rho\sin(\phi)$. Divide both sides by ρ.

$$\cos(\phi) = \sqrt{3}\sin(\phi) \Rightarrow \frac{1}{\sqrt{3}} = \frac{\sin(\phi)}{\cos(\phi)} \Rightarrow \tan(\phi) = \frac{\sqrt{3}}{3} \Rightarrow \phi = \frac{\pi}{6}.$$

(#10) Find the standard form for the plane which contains the points $P\,(0,0,0)$, $Q\,(1,2,-3)$, and $R\,(-2,5,-4)$.

$\overrightarrow{PQ} = \langle 1,2,-3 \rangle \qquad \overrightarrow{PR} = \langle -2,5,-4 \rangle$.

The normal vector for plane will be $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,2,-3 \rangle \times \langle -2,5,-4 \rangle = \langle 7,10,9 \rangle$.

The standard form for the plane is $7\,(x-0)+10\,(y-0)+9\,(z-0) = 0$ or $7x+10y+9z = 0$.

(#11) We have $|\mathbf{r}| = 20$ feet and $|\mathbf{F}| = 16$ pounds. The measure of Angle A does not matter.

If we translate the force vector back to the tail of the lever arm vector, we essentially form a parallelogram.

(a) Since Angle C measures $58°$, the angle of separation must be $180° - 58° = 122°$.

(b) The shortest angular path from the level arm to the force vector is *clockwise*, so by the Right-hand Rule, the torque vector must point INTO the paper.

(c) $|\mathbf{T}| = (20 \text{ feet})\,(16 \text{ lb}) \sin(122°) = 310 * \sin(122°)$ ft-lb.

(#12) True or False? No work is necessary.

(a) If c is a nonzero scalar, then $|c\mathbf{v}|$ always equals $c\,|\mathbf{v}|$.
FALSE. If $c < 0$, then $|c\mathbf{v}| = |c| * |\mathbf{v}|$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ must be parallel.
FALSE. We recall that $\mathbf{u}$ and $\mathbf{v}$ must be perpendicular (orthogonal).

(c) If we have the coordinates $(\rho, \theta, \phi) = \left(2, \frac{\pi}{3}, \frac{\pi}{2}\right)$, then $z = 0$.

TRUE. In spherical coordinates, if $\phi = \frac{\pi}{2}$, then the point must lie in the xy-plane.

(#13) Match the equations with the appropriate shapes below.

(i) $\rho = 7$
This is the sphere. The correct answer is **(a)**.

(ii) $z = -r^2 = -\left(x^2 + y^2\right)$
This is the elliptic (circular) paraboloid which is bowl-shape downward.

(iii) $x^2 - y^2 + z^2 = 4 \Rightarrow \frac{x^2}{4} - \frac{y^2}{4} + \frac{z^2}{4} = 1$

This is the hyperboloid of one sheet because the standard form has ONE negative sign.

(#14) Suppose we have the position function $\mathbf{r}\,(t) = \left\langle t,\ 4 - t^2 \right\rangle$ for $-2 \le t \le 2$.

(a) Sketch the 2-D parabola which is associated with this position function.
This is the parabola, $y = 4 - x^2$, $-2 \le t \le 2$.

(b) Sketch in the position vector $\mathbf{r}\,(1)$ with its tail at the origin.
$\mathbf{r}\,(1) = \left\langle 1,\ 4 - 1^2 \right\rangle = \langle 1, 3 \rangle$.

(c) $\mathbf{v}\,(t) = \langle 1,\ -2t \rangle \Rightarrow \mathbf{v}\,(1) = \langle 1,\ -2 \rangle$.

(d) Sketch in $\mathbf{v}(1)$ so that we can see the geometric relationship between the velocity vector and the path of the object.

The initial point is $(1,3)$ and the terminal point is $(2,1)$. The vector is tangent to the curve and it shows that the object is moving from left to right (and also downward).