Solutions to Assignment #09 – MATH 1401
Spring 2006

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Section 3.3 – Maximum & Minimum

(#12) Find the critical numbers. Classify.

$$\begin{aligned} f(x) &= x^4 + 6x^2 - 2 \\ f'(x) &= 4x^3 + 12x \\ 4x\left(x^2+3\right) &= 0 \end{aligned}$$

Solve:

$$\begin{aligned} 4x^3 + 12x &= 0 \\ 4x\left(x^2+3\right) &= 0. \end{aligned}$$

We have $x = 0$ and $x^2 + 3 = 0$ has no real solutions. So we only have one critical number.

Intervals ⟶ Factors ↓	$(-\infty,\ 0)$	$(0,\ +\infty)$
$4x$	$-$	$+$
x^2+3	$+$	$+$
f'	$-$	$+$

f is decreasing going into $x = 0$ and increasing coming out.

Thus, $x = 0$ must be relative minimum.

(#14) Same.

$$\begin{aligned} f(x) &= \left(x^{2/5} - 3x^{1/5}\right)^2 \\ f'(x) &= 2\left(x^{2/5} - 3x^{1/5}\right)\left(\frac{2}{5}x^{-3/5} - \frac{3}{5}x^{-4/5}\right) \\ &= 2x^{1/5}\left(x^{1/5} - 3\right)\left(\frac{1}{5}x^{-4/5}\right)\left(2x^{1/5} - 3\right) \\ &= \frac{2}{5}\left(x^{1/5}\right)\left(x^{-4/5}\right)\left(x^{1/5} - 3\right)\left(2x^{1/5} - 3\right). \end{aligned}$$

Solve by factoring out $x^{1/5}$.

$$x^{1/5}\left(x^{1/5}-3\right)=0$$

$x^{1/5}=\sqrt[5]{x}=0 \Rightarrow x=0$.

We notice that f' has a $x^{-3/5}$ term in it, so f' is undefined at $x=0$. [Sharp turn.]

$\sqrt[5]{x}=3 \Rightarrow x=3^5=243$.

Factor out $\frac{1}{5}$ and multiply everything by $x^{4/5}$ (or factor out $x^{-4/5}$).

$$\begin{aligned}\frac{2}{5}x^{-3/5}-\frac{3}{5}x^{-4/5} &= 0\\ \left(\frac{1}{5}x^{-4/5}\right)\left(2x^{-3/5}x^{4/5}-3x^{-4/5}x^{4/5}\right) &= 0\\ \left(\frac{1}{5}x^{-4/5}\right)\left(2x^{1/5}-3\right) &= 0.\end{aligned}$$

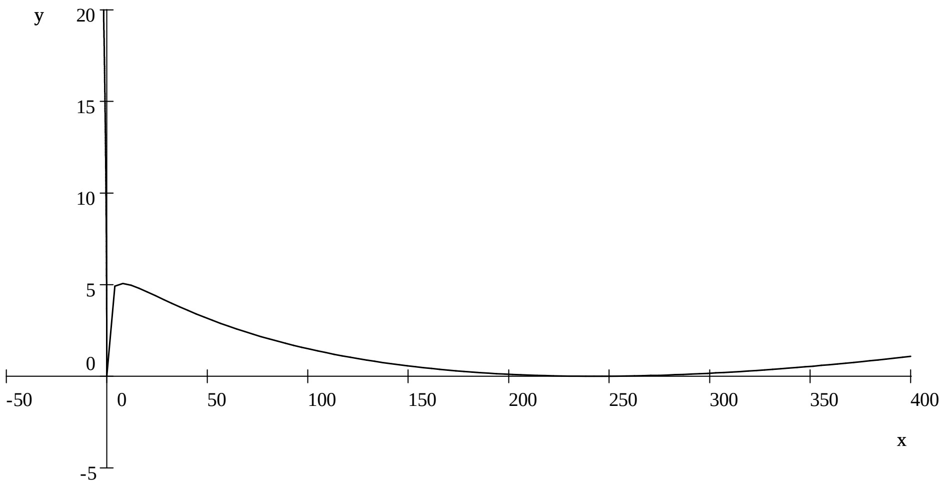
We know that $x^{-4/5}=\dfrac{1}{x^{4/5}}$ can never equal zero (numerator is one).

Solve:

$$2x^{1/5}-3=0 \Rightarrow x^{1/5}=\frac{3}{2} \Rightarrow x=\left(\frac{3}{2}\right)^5=\frac{243}{32}.$$

Intervals $\longrightarrow$ Factors $\downarrow$	$(-\infty,\ 0)$	$\left(0,\ \frac{243}{32}\right)$	$\left(\frac{243}{32},\ 243\right)$	$(243,\ +\infty)$
$\frac{2}{5}\left(x^{1/5}\right)$	$-$	$+$	$+$	$+$
$x^{-4/5}$	$+$	$+$	$+$	$+$
$x^{1/5}-3$	$-$	$-$	$-$	$+$
$2x^{1/5}-3$	$-$	$-$	$+$	$+$
f'	$-$	$+$	$-$	$+$

We have relative minimum at $x=0$, relative maximum at $x=\dfrac{243}{32}$, and a relative minimum at $x=243$.



(#20) Quotient Rule.

$$
\begin{aligned}
f(x) &= \frac{x^2 - x + 4}{x - 1} \\
f'(x) &= \frac{(x-1)\left[x^2 - x + 4\right]' - \left(x^2 - x + 4\right)[x-1]'}{(x-1)^2} \\
&= \frac{(x-1)(2x-1) - \left(x^2 - x + 4\right)(1)}{(x-1)^2} \\
&= \frac{x^2 - 2x - 3}{(x-1)^2} = \frac{(x+1)(x-3)}{(x-1)^2} = 0.
\end{aligned}
$$

We set the numerator equal to zero. We have $x = -1,\ 3$.

We note that denominator is always positive and the function itself is not defined at $x = 1$ (vertical asymptote).

Intervals $\longrightarrow$ Factors $\downarrow$	$(-\infty,\ -1)$	$(-1,\ 1)$	$(1,\ 3)$	$(3,\ +\infty)$
$x+1$	$-$	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
f'	$+$	$-$	$-$	$+$

We see that $x = -1$ is a relative maximum and $x = 3$ is a relative minimum.

We can also determine that

$$
\lim_{x \to 1^-} \frac{x^2 - x + 4}{x - 1} = -\infty
$$

since f is decreasing as it approaches the vertical asymptote from the left. Similarly, we know that

$$\lim_{x\to 1^+} \frac{x^2 - x + 4}{x - 1} = +\infty$$

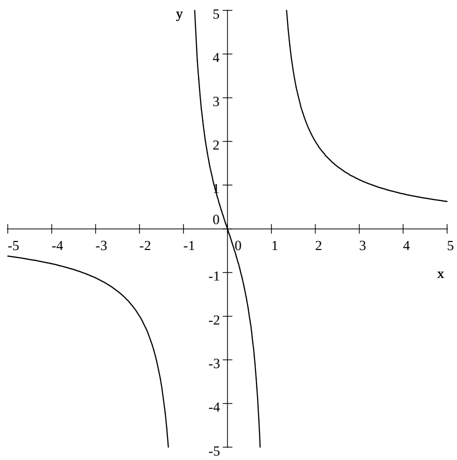
because f is decreasing and it must have started at $+\infty$ on the right side of the vertical asymptote.

(#22) Two vertical asymptotes at $x = \pm 1$.

$$\begin{aligned} f(x) &= \frac{3x}{x^2 - 1} \\ f'(x) &= -\frac{3\left(x^2+1\right)}{\left(x^2-1\right)^2} = 0. \end{aligned}$$

Solve: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow$ No real solutions.

No critical points. No relative extrema.

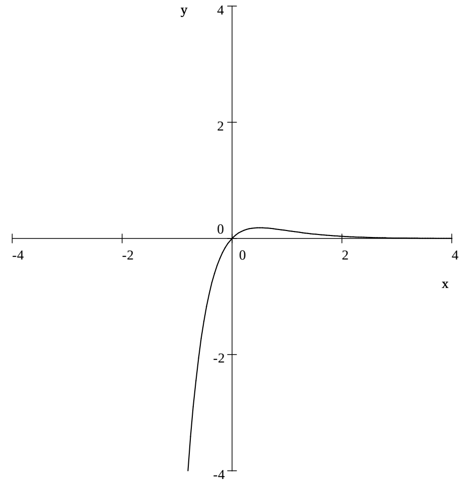


(#24) Product Rule.

$$\begin{aligned} f(x) &= xe^{-2x} \\ f'(x) &= x\left[e^{-2x}\right]' + e^{-2x}\left[x\right]' = x\left(-2e^{-2x}\right) + e^{-2x} \\ &= \left(-2x+1\right)e^{-2x}. \end{aligned}$$

We recall that e to the anything is *never* zero. The only critical number is clearly $x = \frac{1}{2}$.

Intervals ⟶ Factors ↓	$\left(-\infty,\ \frac{1}{2}\right)$	$\left(\frac{1}{2},\ +\infty\right)$
$-2x+1$	$+$	$-$
e^{-2x}	$+$	$+$
f'	$+$	$-$



We see that $x = \frac{1}{2}$ must be a relative maximum.

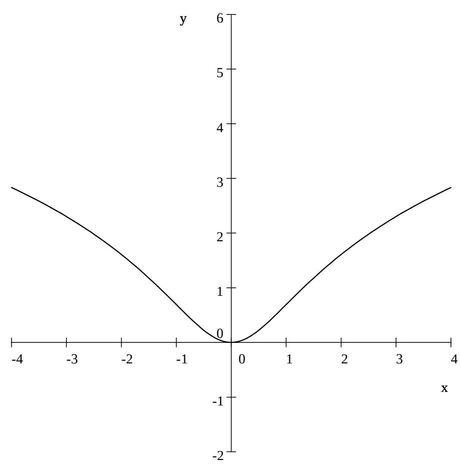
(#30) Natural log. The domain is all real numbers.

$$f(x) = \ln\left(x^2+1\right)$$

$$f'(x) = \frac{2x}{x^2+1} = 0.$$

The denominator is always positive. The only critical number is $x = 0$.

Intervals ⟶ Factors ↓	$(-\infty,\ 0)$	$(0,\ +\infty)$
$2x$	$-$	$+$
x^2+1	$+$	$+$
f'	$-$	$+$



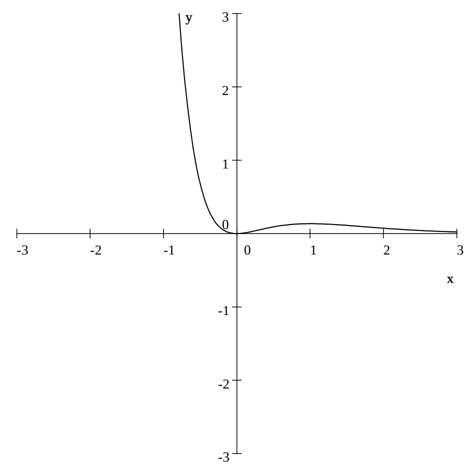
Section 3.4

(#14) We basically use the same tools that I introduced in Section 3.3. Product Rule.

$$\begin{aligned} f(x) &= x^2 e^{-2x} \\ f'(x) &= x^2\left(-2e^{-2x}\right) + e^{-2x}(2x) \\ &= -2x(x-1)e^{-2x} \end{aligned}$$

The critical numbers are $x = 0,\ 1$.

Intervals $\longrightarrow$ / Factors $\downarrow$	$(-\infty,\ 0)$	$(0,\ 1)$	$(1,\ +\infty)$
$-2x$	$+$	$-$	$-$
$x-1$	$-$	$-$	$+$
e^{-2x}	$+$	$+$	$+$
f'	$-$	$+$	$-$



We see that $x=0$ is a relative minimum and $x=1$ is a relative maximum.

(#16) Look at the domain.

$$f(x)=\sin^{-1}\left(1-\frac{1}{x^2}\right)$$

The input to the inverse sine must be between (-1) and 1.

$$\begin{aligned}-1 &\le 1-\frac{1}{x^2}\le 1\\ -1-1 &\le 1-1-\frac{1}{x^2}\le 1-1\\ -2 &\le -\frac{1}{x^2}\le 0\end{aligned}$$

Since $x^2\ge 0$, we can multiply everything by x^2 and not change the inequalities.

$$-2x^2\le -1\le 0.$$

We know that $-1\le 0$, so the last inequality is useless.

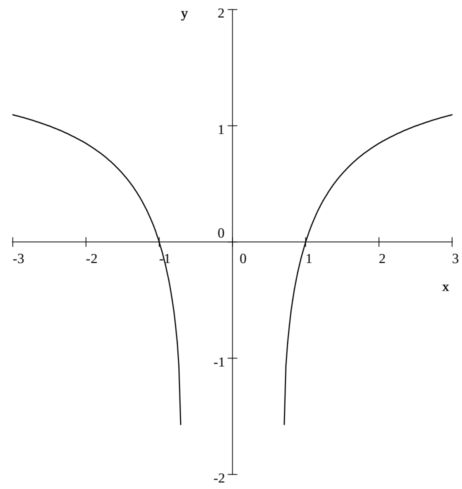
$$-2x^2\le -1\Rightarrow x^2\ge\frac{1}{2}.$$

So the domain is $x\ge\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$ or $x\le-\frac{\sqrt{2}}{2}$.

Find the critical numbers.

$$\begin{aligned}f'(x) &= \frac{1}{\sqrt{1-\left(1-\frac{1}{x^2}\right)^2}}\left[1-\frac{1}{x^2}\right]'=\frac{1}{\sqrt{1-\left(1-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\left(\frac{2}{x^3}\right)\\ &= \frac{1}{\sqrt{\frac{2}{x^2}-\frac{1}{x^4}}}\left(\frac{2}{x^3}\right)=\frac{1}{\sqrt{\frac{2x^2-1}{x^4}}}\left(\frac{2}{x^3}\right)=\frac{1}{\left(\frac{\sqrt{2x^2-1}}{x^2}\right)}\left(\frac{2}{x^3}\right)\\ &= \frac{2}{\left(\frac{x^3\sqrt{2x^2-1}}{x^2}\right)}=\frac{2}{x\sqrt{2x^2-1}}.\end{aligned}$$

No critical numbers since the numerator can never equal zero. We see that $x = 0$ is not in the domain.



(#40) Sketch a graph. We assume that f is continuous.

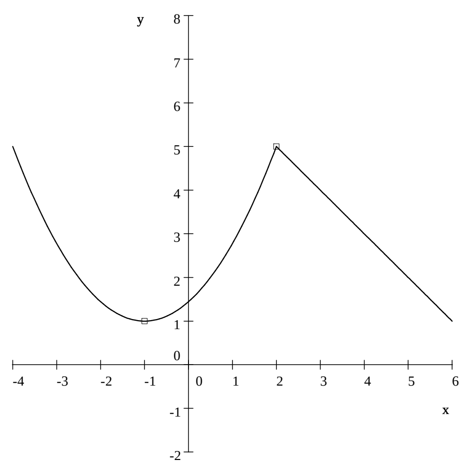
$f(-1) = 1,\ f(2) = 5.$

$f'(-1) = 0 \Rightarrow x = -1$ is a critical number.

$f'(2)$ does not exist. Either $x = 2$ has a vertical tangent or a sharp turn (or possibly both).

Intervals $\longrightarrow$ Factors $\downarrow$	$(-\infty,\ -1)$	$(-1,\ 2)$	$(2,\ +\infty)$
f'	$-$	$+$	$-$

We see that $x = -1$ must be a relative minimum. There is a relative maximum at $x = 2$, but $f'(2)$ does not exist so it's probably a sharp turn.



Section 3.5

(#35) Determine all significant features. That includes concavity and inflection points.

$$f(x) = x^2\sqrt{x^2-9}$$

Look at the domain. We must have

$$x^2 - 9 \geq 0 \Rightarrow x \leq -3 \ \text{ or } \ x \geq 3.$$

Find the critical numbers.

$$\begin{aligned} f'(x) &= x^2\left[\left(x^2-9\right)^{1/2}\right]' + \sqrt{x^2-9}\left[x^2\right]' = x^2\left(\frac{1}{2}\left(x^2-9\right)^{-1/2}(2x)\right) + 2x\sqrt{x^2-9} \\ &= \frac{x^3}{\sqrt{x^2-9}} + 2x\sqrt{x^2-9} = 0. \end{aligned}$$

Multiply through by $\sqrt{x^2-9}$.

$$\begin{aligned} x^3 + 2x\sqrt{x^2-9}\sqrt{x^2-9} &= 0 * \sqrt{x^2-9} \\ x^3 + 2x\left(x^2-9\right) &= 0 \\ x^3 + 2x^3 - 18x &= 0 \\ 3x^3 - 18x &= 0 \\ 3x\left(x^2-6\right) &= 0. \end{aligned}$$

We have $x = 0, \ \pm\sqrt{6}$ as candidates, but NONE of them are in the domain! So f has not critical numbers.

Find the inflection points.

$$\begin{aligned} f''(x) &= \left[\frac{x^3}{\sqrt{x^2-9}}\right]' + 2\left[x\sqrt{x^2-9}\right]' \\ &= \frac{\sqrt{x^2-9}\left[x^3\right]' - x^3\left[\left(x^2-9\right)^{1/2}\right]'}{\left(\sqrt{x^2-9}\right)^2} + 2\left(x\left[\left(x^2-9\right)^{1/2}\right]' + \sqrt{x^2-9} * [x]'\right) \\ &= \frac{\sqrt{x^2-9}\left(3x^2\right) - x^3\left(\frac{1}{2}\right)\left(x^2-9\right)^{-1/2}(2x)}{x^2-9} + 2\left(x\left(\frac{1}{2}\right)\left(x^2-9\right)^{-1/2}(2x) + \sqrt{x^2-9}\right) \\ &= \frac{3x^2\sqrt{x^2-9} - \frac{x^4}{\sqrt{x^2-9}}}{x^2-9} + 2\left(\frac{x^2}{\sqrt{x^2-9}} + \sqrt{x^2-9}\right) \\ &= \frac{\left(\frac{3x^2\sqrt{x^2-9}\sqrt{x^2-9}-x^4}{\sqrt{x^2-9}}\right)}{x^2-9} + 2\left(\frac{x^2+\sqrt{x^2-9}\sqrt{x^2-9}}{\sqrt{x^2-9}}\right) \\ &= \frac{3x^2\left(x^2-9\right)-x^4}{\left(x^2-9\right)^{3/2}} + 2\left(\frac{2x^2-9}{\left(x^2-9\right)^{1/2}}\right) = \frac{3x^4-27x^2-x^4}{\left(x^2-9\right)^{3/2}} + \frac{4x^2-18}{\left(x^2-9\right)^{1/2}}. \end{aligned}$$

The LCD is $\left(x^2-9\right)^{3/2}$. We can modify and combine the fractions.

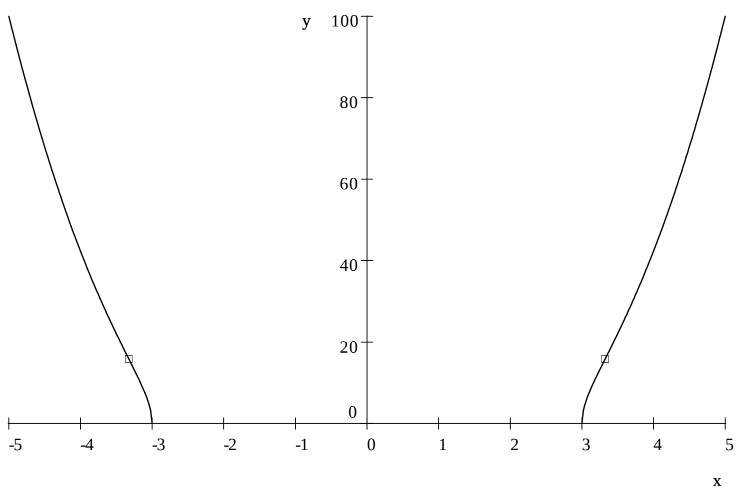
$$\begin{aligned} f''(x) &= \frac{3x^4-27x^2-x^4}{\left(x^2-9\right)^{3/2}} + \frac{\left(4x^2-18\right)\left(x^2-9\right)}{\left(x^2-9\right)^{1/2}\left(x^2-9\right)} \\ &= \frac{3x^4-27x^2-x^4+4x^4-54x^2+162}{\left(x^2-9\right)^{3/2}} \\ &= \frac{6x^4-81x^2+162}{\left(x^2-9\right)^{3/2}} = 0. \end{aligned}$$

The quartic polynomial in the numerator is only solvable through the Quartic Formula which we don't know. Technology gives us approximate solutions.

$$x \doteq \pm 1.5626, \ \pm 3.3254.$$

The first two candidates are not in the domain, so the inflection points must occur at $x \doteq \pm 3.3254$.

I placed markers at the inflection points.

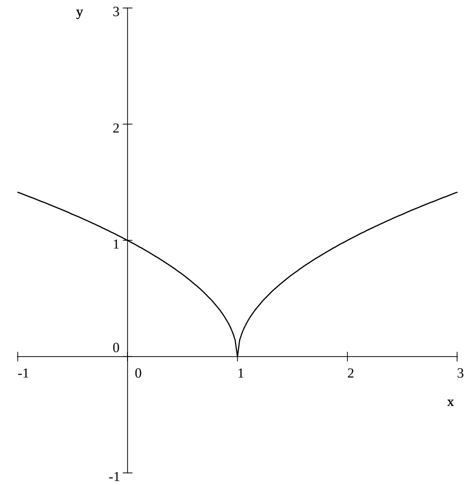


(#52) Sketch a continuous curve.

$f(1)=0 \Rightarrow x=1$ is a critical number.

Intervals $\longrightarrow$	$(-\infty,\ 1)$	$(1,\ +\infty)$	$(1,\ +\infty)$
f'	$-$	$+$	$-$
f''	$-$	$-$	$+$
e^{-2x}	$+$	$+$	$+$

This says that $x = 1$ must be a relative minimum, but we see that on both sides of $x = 1$, the curve is concave DOWN, not concave up. Thus, it must be a sharp turn. We note that there is no requirement for $f'(1)$ or $f''(1)$ to exist.



(#74) Suppose that $T(t)$ is a sick person's temperature at time t.

Which would be better news at time $t = 0$:

(a) $T''(0) = 2$ or

(b) $T''(0) = -2$ or

(c) would you need to know the value of $T'(0)$ and $T(0)$ to determine which is better?

The answer is that the knowledge of concavity is certainly not enough to judge the news.

If the person's original temperature were below normal $(T(0)$ is below normal$)$, then either concavity could be bad news.

If $T''(0) = 2$, then T could still be dropping because T has not reached its local minimum. So the patient could still die (too cold).

If $T''(0) = -2$, then T could be decreasing in an accelerated manner and the patient could die quickly.

So we need all of the information to predict the future: $T(0)$, $T'(0)$, and $T''(0)$.

Knowledge of all three quantities allows us to create a quadratic model of temperature vs. time, much like our linear approximation from the previous sections.