

Solutions to Assignment #07 – MATH 1401
Spring 2006

Kawai/Lashuk/Acks

Section 2.9

- (#4) For this exercise, decide if Rolle's Theorem or the Mean Value Theorem can be applied. If all of the hypotheses for the particular theorem can be satisfied, then find the value of $x = c$ which satisfies the conclusion. Graph the appropriate tangent line!

$$f(x) = x^3 + x^2 \text{ on } [-1, 1]$$

This is a cubic polynomial, so we know that it is differentiable on any interval.

We have $f(-1) = 0$ and $f(1) = 2$, so it's not Rolle's Theorem.

We would need $f(a) = f(b) = 0$. It must be the MVT.

The MVT says that there exists $x = c$ in $(-1, 1)$ such that

$$\begin{aligned} f'(c) &= \text{Average Rate of Change on that interval} \\ &= \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1. \end{aligned}$$

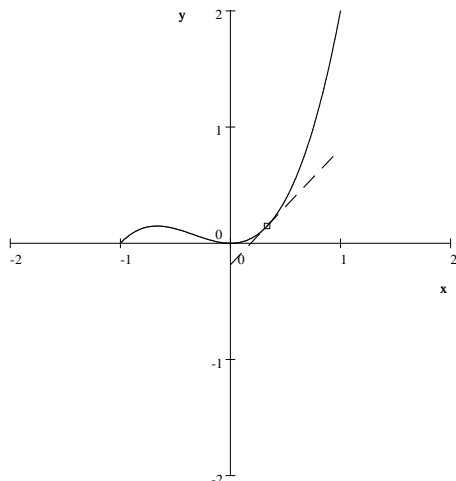
We must find an $x = c$ in that interval where $f'(c) = 1$.

$$\begin{aligned} f'(x) &= 3x^2 + 2x = 1 \\ 3x^2 + 2x - 1 &= 0 \\ (3x - 1)(x + 1) &= 0 \end{aligned}$$

$$x = \frac{1}{3}, -1.$$

We want $x = \frac{1}{3}$. Since $f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 = \frac{4}{27}$, the equation of the tangent line is

$$y = 1\left(x - \frac{1}{3}\right) + \frac{4}{27}.$$



In this case, there is only one value of $x = c$ which satisfies the MVT.

(#6) Same.

$$f(x) = \sin(x) \text{ on } [-\pi, 0].$$

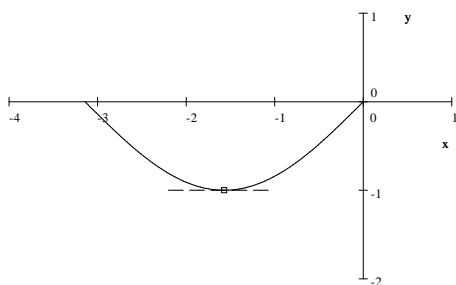
The sine function is continuous and differentiable everywhere.

We have $f(-\pi) = 0$ and $f(0) = 0$, so Rolle's Theorem should apply.

We must find an $x = c$ in $(-\pi, 0)$ such that $f'(c) = 0$.

$$f'(x) = \cos(x) = 0.$$

The correct value is $x = -\frac{\pi}{2}$. Since $f\left(-\frac{\pi}{2}\right) = -1$, the equation of the tangent line must be $y = -1$.



(#10) Determine if the function is increasing, decreasing, or neither.

$$\begin{aligned} f(x) &= x^5 + 3x^3 - 1 \\ f'(x) &= 5x^4 + 9x^2. \end{aligned}$$

Since $f'(x)$ is the sum of even powers, we know that x^4 and x^2 are always nonnegative.

Thus, we see that $f'(x) = 0$ when $x = 0$, but $f'(x)$ is always positive everywhere else. Thus, we conclude that f is increasing.

(#24) Prove that a third-degree polynomial ($f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$) has at most 3 zeros.

Suppose it had four (or more) zeros. Without loss of generality, we would have four values of x (let's call them $x_1 < x_2 < x_3 < x_4$) such that

$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0.$$

From our discussion in lecture, we invoke Rolle's Theorem.

On the interval (x_1, x_2) , there must be an $x = c$ such that $f'(c) = 0$.

The same thing must be true on the intervals (x_2, x_3) and (x_3, x_4) . Thus, the function $f'(x)$ must have THREE zeros.

If $f(x) = ax^3 + bx^2 + cx + d$, then

$$f'(x) = 3ax^2 + 2bx + c.$$

This is a quadratic equation and we know from the Quadratic Formula, this can have at most TWO zeros. So there's the contradiction. Our original supposition that f has four zeros cannot be correct, and by the same argument, we see that at most, f can have three zeros.

Section 3.1

(#4) Find the linear approximation (the tangent line) at $x = 1$.

The graph of $f(x) = \frac{2}{x}$ is technically a hyperbola (rectilinear hyperbola, since its asymptotes are parallel to the coordinate axes).

$$f'(x) = 2[x^{-1}]' = 2(-1x^{-2}) = -\frac{2}{x^2}.$$

When $x = 1$, the slope is $f'(1) = -2$. So the equation of the tangent line is

$$y = -2(x - 1) + 2.$$

So when x is close to $x_0 = 1$, we can use this approximation rather than performing the lengthier (more exact) calculation.

If we use this to approximate the value of $\frac{2}{1.01}$, then we have $x = 1.01$, and

$$f(1.01) \approx -2(1.01 - 1) + 2 = -2(0.01) + 2 = 1.98.$$

We compare this with the real value

$$\frac{2}{1.01} \doteq 1.9802$$

and we see that our linear approximation is very close to the “real thing”.

Remember that this only works when x is very close to x_0 .

(#12) Estimate $\sqrt[4]{16.08}$ using the appropriate linear approximation.

We must choose $f(x) = \sqrt[4]{x} = x^{1/4}$ and we see that, conveniently, $f(16) = 16^{1/4} = 2$.

In order for this approximation to be halfway useful, $f(x_0)$ must be easy to calculate, so we choose $x_0 = 16$.

$$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}.$$

When $x_0 = 16$, we have

$$f'(x_0) = \frac{1}{4(16^{3/4})} = \frac{1}{4((16^{1/4})^3)} = \frac{1}{4(2^3)} = \frac{1}{32}.$$

The linear approximation there will be

$$y = \frac{1}{32}(x - 16) + 2.$$

We want the approximation when $x = 16.08$.

$$\sqrt[4]{16.08} \approx \frac{1}{32}(16.08 - 16) + 2 = \frac{0.08}{32} + 2 = \frac{0.01}{4} + 2 = 2.0025.$$

We compare this to the exact approximation $\sqrt[4]{16.08} \doteq 2.002495$. Our linear approximation is very close!

(#66) When x is a small number close to zero (positive or negative), we say that

$$e^x \approx 1 + x.$$

Show that Planck's Law

$$f(x) = \frac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}$$

can be approximated by the expression

$$f(x) \approx \frac{8\pi kT}{x^4}.$$

This is known as the Rayleigh-Jeans Law.

So look at the denominator of Planck's Law. If the quantity $\frac{hc}{kTx}$ is a small number close to zero, then we have

$$e^{hc/(kTx)} \approx 1 + \frac{hc}{kTx}.$$

It turns out that h is a really small multiplicative constant ($6.6 * 10^{-34}$ in the appropriate metric units).

This gives us

$$f(x) \approx \frac{8\pi hcx^{-5}}{\left(1 + \frac{hc}{kTx}\right) - 1} = \frac{8\pi hcx^{-5}}{\left(\frac{hc}{kTx}\right)} = \left(\frac{8\pi hcx^{-5}}{1}\right) \left(\frac{kTx}{hc}\right) = 8\pi kTx^{-4} = \frac{8\pi kT}{x^4}. \checkmark$$