

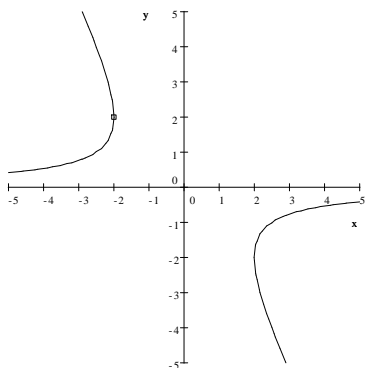
**Solutions to Assignment #06 – MATH 1401**  
Spring 2006

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**Section 2.8**

(#4) Compute the slope of the tangent line at  $(-2, 2)$  if

$$y^2 + 2xy + 4 = 0.$$



It's a hyperbola.

It appears that the tangent line may be vertical!  
If this is the case, then the derivative is undefined! Let's see what happens...

(a) First, solve for  $y$  explicitly and then take the derivative. We will need the Quadratic Formula.

$$y^2 + 2xy + 4 = 0$$

We have  $a = 1$ ,  $b = 2x$ , and  $c = 4$ .

$$\begin{aligned} y &= \frac{-2x \pm \sqrt{(2x)^2 - 4(1)(4)}}{2(1)} = \frac{-2x \pm \sqrt{4x^2 - 16}}{2} = \frac{-2x \pm \sqrt{4(x^2 - 4)}}{2} \\ &= \frac{-2x \pm 2\sqrt{x^2 - 4}}{2} = -x \pm \sqrt{x^2 - 4} = y. \end{aligned}$$

When  $x = -2$ , we have

$$y = -(-2) \pm \sqrt{(-2)^2 - 4} = 2 \pm \sqrt{0} = 2.$$

Interesting. It doesn't matter if we use  $y = -x + \sqrt{x^2 - 4}$  or  $y = -x - \sqrt{x^2 - 4}$ . Both branches meet at  $(-2, 2)$ . We will choose

$$y = -x + \sqrt{x^2 - 4} = -x + (x^2 - 4)^{1/2}$$

and then find  $y'$ .

$$y' = -1 + \frac{1}{2} (x^2 - 4)^{-1/2} (2x) = -1 + \frac{x}{\sqrt{x^2 - 4}}.$$

We see that this is undefined when  $x = -2$ , so we can guess that the tangent line is vertical and the equation is  $x = -2$ .

[Verify that if we had used  $y = -x - (x^2 - 4)^{1/2}$ , then we would have obtained the same result!]

(b) Second, find  $y'$  using implicit differentiation.

$$\frac{d}{dx} [y^2] + 2\frac{d}{dx} [xy] + \frac{d}{dx} [4] = \frac{d}{dx} [0]$$

$$2yy' + 2(xy' + y) + 0 = 0$$

$$2yy' + 2xy' + 2y = 0$$

$$yy' + xy' + y = 0$$

$$yy' + xy' = -y$$

$$(x + y)y' = -y$$

$$y' = \frac{-y}{x + y}.$$

We note that  $\frac{d}{dx} [xy]$  requires the Product Rule. On Line #3, we divided everything by 2.

When  $x = -2$  and  $y = 2$ , the denominator is zero and  $y'$  is undefined. This supports the graphical evidence that the tangent line is the vertical line  $x = -2$ .

(#10) Find  $y'$  implicitly.

$$3x + y^3 - 4y = 10x^2$$

$$\frac{d}{dx} [3x] + \frac{d}{dx} [y^3] - 4\frac{d}{dx} [y] = 10\frac{d}{dx} [x^2]$$

$$3 + 3y^2y' - 4y' = 10(2x)$$

$$(3y^2 - 4)y' = 20x - 3$$

$$y' = \frac{20x - 3}{3y^2 - 4}.$$

(#16) Same.

$$e^{x^2}y - 3y = x^2 + 1$$

$$\frac{d}{dx} [e^{x^2}y] - 3\frac{d}{dx} [y] = \frac{d}{dx} [x^2 + 1]$$

$$e^{x^2}y' + y\frac{d}{dx} [e^{x^2}] - 3y' = 2x$$

$$e^{x^2}y' + ye^{x^2}(2x) - 3y' = 2x$$

$$(e^{x^2} - 3)y' = 2x - 2xye^{x^2}$$

$$y' = \frac{2x - 2xye^{x^2}}{e^{x^2} - 3} = \frac{2x(1 - ye^{x^2})}{e^{x^2} - 3}.$$

(#26) Find the  $(x, y)$  locations of all horizontal and vertical tangent lines. We want  $y'$  to equal zero or  $y'$  to be undefined.

$$xy^2 - 2y = 2$$

$$\frac{d}{dx} [xy^2] - 2 \frac{d}{dx} [y] = \frac{d}{dx} [2]$$

$$\begin{aligned} x(2yy') + y^2 - 2y' &= 0 \\ 2xyy' - 2y' &= -y^2 \\ (2xy - 2)y' &= -y^2 \end{aligned}$$

$$y' = -\frac{y^2}{2xy - 2} = -\frac{y^2}{2(xy - 1)}$$

This expression is certainly equal to zero when  $y = 0$  (on the x-axis). So if we can find the locations where the original curve intersects the x-axis, we will have all the horizontal tangents. Substitute  $y = 0$  into the original equation.

$$\begin{aligned} x(0^2) - 2(0) &= 2 \\ 0 &= 2. \text{ No solution!} \end{aligned}$$

The original curve does not have any horizontal tangent lines!

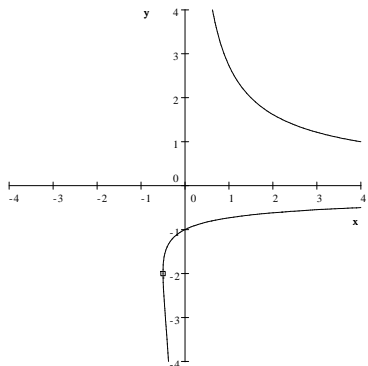
Can the expression for  $y'$  be undefined? Possibly. We would need to have  $xy = 1$ .

Substitute this back into the original equation.

$$\begin{aligned} xy^2 - 2y &= 2 \\ (xy)y - 2y &= 2 \\ 1y - 2y &= 2 \\ y &= -2 \end{aligned}$$

Since we're assuming that  $xy = 1$  and we have  $y = -2$ , the value of  $x$  must be  $x = -\frac{1}{2}$ .

We believe that there is a vertical tangent line at  $\left(-\frac{1}{2}, -2\right)$ . Here's the graph.



The graphical evidence supports our

conclusion. The vertical line  $x = -\frac{1}{2}$

should be a vertical tangent line.

(#32) See handout. The result of the first implicit differentiation was

$$e^{xy} (y + xy') + 2y' - 3 = (\cos(y)) y'.$$

The Product Rule was applied to  $\frac{d}{dx} [e^{xy}] = e^{xy} \frac{d}{dx} [xy] = e^{xy} (xy' + x)$ .

We can solve for  $y'$ . The result was

$$y' = \frac{3 - ye^{xy}}{xe^{xy} - \cos(y) + 2}.$$

It turned out that our best strategy was to go ahead and differentiate the first equation! It allows us to avoid the Quotient Rule, which always makes the calculations worse!

$$\frac{d}{dx} [e^{xy} (y + xy')] + 2\frac{d}{dx} [y'] - \frac{d}{dx} [3] = \frac{d}{dx} [(\cos(y)) y']$$

$$\frac{d}{dx} [e^{xy} (y + xy')] + 2\frac{d}{dx} [y'] - \frac{d}{dx} [3] = \cos(y) y'' + y' (-\sin(y) y')$$

$$e^{xy} \frac{d}{dx} [y + xy'] + (y + xy') \frac{d}{dx} [e^{xy}] + 2y'' = \cos(y) y'' - \sin(y) (y')^2$$

$$e^{xy} (y' + xy'' + y') + (y + xy') (e^{xy} (xy' + y)) + 2y'' = \cos(y) y'' - \sin(y) (y')^2$$

$$e^{xy} (xy'' + 2y') + e^{xy} (xy' + y)^2 + 2y'' = \cos(y) y'' - \sin(y) (y')^2$$

$$xe^{xy} y'' + 2e^{xy} y' + e^{xy} (xy' + y)^2 + 2y'' = \cos(y) y'' - \sin(y) (y')^2$$

$$xe^{xy} y'' - \cos(y) y'' + 2y'' = -\sin(y) (y')^2 - e^{xy} (xy' + y)^2 - 2e^{xy} y'$$

$$(xe^{xy} - \cos(y) + 2) y'' = -\left(\sin(y) (y')^2 + e^{xy} (xy' + y)^2 + 2e^{xy} y'\right)$$

$$y'' = -\left(\frac{\sin(y) (y')^2 + e^{xy} (xy' + y)^2 + 2e^{xy} y'}{xe^{xy} - \cos(y) + 2}\right).$$

I would leave it this way. Check out the handout if you want to try and substitute in the formula for  $y'$ .

(#44) Find the derivatives. We know that  $[\sin^{-1}(x)]' = \frac{1}{\sqrt{1-x^2}}$ .

$$[\sin^{-1}(x^3 + 1)]' = \frac{1}{\sqrt{1 - (x^3 + 1)^2}} * \frac{d}{dx} [x^3 + 1] = \frac{3x^2}{\sqrt{1 - (x^3 + 1)^2}}.$$

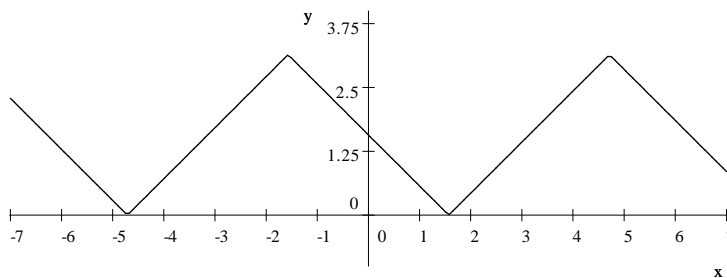
(#46) Find the derivatives. We know that  $[\cos^{-1}(x)]' = -\frac{1}{\sqrt{1-x^2}}$ .

$$[\cos^{-1}(\sin(x))]' = -\frac{1}{\sqrt{1-(\sin(x))^2}} * \frac{d}{dx}[\sin(x)] = -\frac{\cos(x)}{\sqrt{\cos^2(x)}}$$

Since  $x$  can be any angle, the value of  $\cos(x)$  could be positive or negative. Thus, we say  $\sqrt{\cos^2(x)} = |\cos(x)|$ .

$$\begin{aligned} [\cos^{-1}(\sin(x))]' &= -\frac{\cos(x)}{|\cos(x)|} \\ &= \begin{cases} -1, & \cos(x) > 0 \\ +1, & \cos(x) < 0 \\ \text{undef}, & \cos(x) = 0 \end{cases} \end{aligned}$$

Here's the graph.



We see sharp turns at

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc.}$$

Between the sharp turns the slopes are either (+1) or (-1).

(#50) General Power Rule first.

$$\begin{aligned} [(2 + \tan^{-1}(x))^{1/2}]' &= \frac{1}{2} (2 + \tan^{-1}(x))^{-1/2} * [2 + \tan^{-1}(x)]' \\ &= \frac{1}{2\sqrt{2 + \tan^{-1}(x)}} \left( \frac{1}{1+x^2} \right) = \frac{1}{2(1+x^2)\sqrt{2 + \tan^{-1}(x)}}. \end{aligned}$$

(#52) Quotient Rule.

$$\begin{aligned} \left[ \frac{\tan^{-1}(x)}{x^2 + 1} \right]' &= \frac{(x^2 + 1)[\tan^{-1}(x)]' - \tan^{-1}(x)[x^2 + 1]'}{(x^2 + 1)^2} \\ &= \frac{(x^2 + 1)\left(\frac{1}{1+x^2}\right) - 2x \tan^{-1}(x)}{(x^2 + 1)^2} \\ &= \frac{1 - 2x \tan^{-1}(x)}{(x^2 + 1)^2}. \end{aligned}$$

(#56) From our discussion in lecture, we saw that

$$\frac{d\theta}{dt} = -\frac{260}{4 + D^2}.$$

So we can find the angular velocity for any given value of  $D$ . When  $\frac{d\theta}{dt} = -3$ , we have

$$-3 = -\frac{260}{4 + D^2} \Rightarrow 12 + 3D^2 = 260 \Rightarrow 3D^2 = 248 \Rightarrow D^2 = \frac{248}{3}$$

$$D = \sqrt{\frac{248}{3}} \doteq 9.09 \text{ feet.}$$

The ball travels a total distance of 60.5 feet. The batter watches the ball for the first 30 feet or so and then decides to swing (or not swing). If the batter is in the process of swinging, then he has about 21 more feet of trajectory to make small adjustments in his swing. During the last 9 feet, the batter cannot track the ball with his eyes.

All of this happens in about 0.5 seconds!