Soutions to Assignment #05 – MATH 1401

Spring 2006

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This counted as Quiz #05!

Section 2.6

(#30) Find an equation of the tangent line to $y = f(x) = x\sin(x)$ at $x = \frac{\pi}{2}$.

We need the derivative and the Product Rule.

$$[x\sin(x)]' = x[\sin(x)]' + \sin(x)[x]' = x\cos(x) + \sin(x).$$

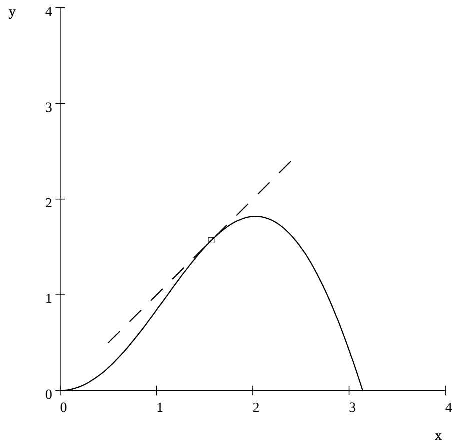
When $x = \frac{\pi}{2}$, we have

$$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 0 + 1 = 1.$$

The point on the curve is $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, so the tangent line is

$$y = 1\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}.$$

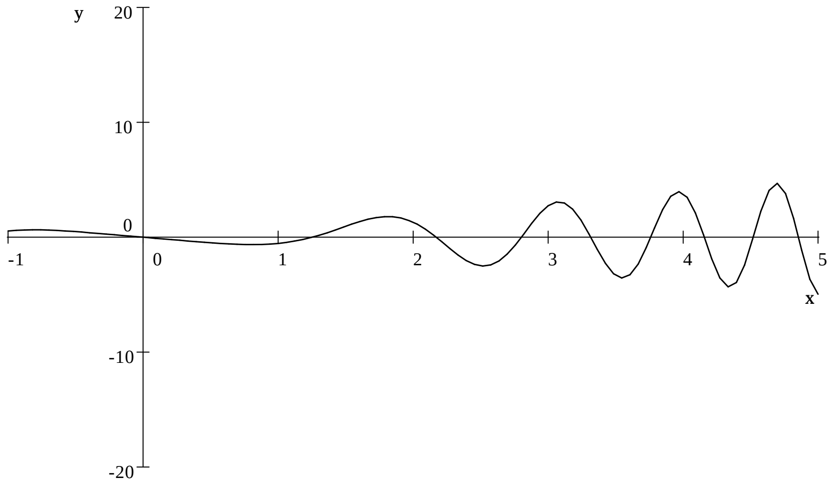
Here's a sketch.

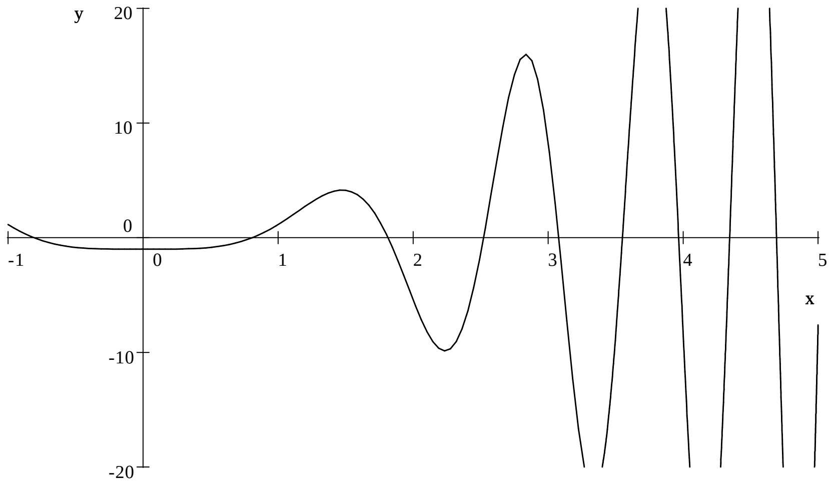


(#32) Use the position function $s(t) = t\cos\left(t^2 + \pi\right)$ to evaluate the velocity function at $t = 0$.

$$\begin{aligned} v(t) &= s'(t) = t\left[\cos\left(t^2+\pi\right)\right]' + \cos\left(t^2+\pi\right)[t]' \\ &= t\left(-\sin\left(t^2+\pi\right)\left[t^2+\pi\right]'\right) + \cos\left(t^2+\pi\right)(1) \\ &= t\left(-\sin\left(t^2+\pi\right)(2t)\right) + \cos\left(t^2+\pi\right) \\ &= -2t^2\sin\left(t^2+\pi\right) + \cos\left(t^2+\pi\right). \end{aligned}$$

You might want to compare the graphs of $s(t)$ [first one] with $v(t)$ [second one].





In the graph of $y = s(t)$, we see the amplitude of the humps increasing, but more importantly, the humps are being compressed (horizontally) and this is causing $y = s'(t) = v(t)$ to oscillate wildly. The slopes in $s(t)$ are steadily becoming steeper and thus, $v(t)$ increases in magnitude (in both directions).

(#36) I hope you looked at (#35) first.

So we know that

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \end{aligned}$$

is the parameterization of the unit circle. If we let $0 \leq t \leq 2\pi$, then our position start at $(1, 0)$ when $t = 0$, and then we trace out the unit circle counterclockwise, until we return to $(1, 0)$ at $t = 2\pi$.

Explain the meanings of the constants a, b, and c in the general model

$$\begin{aligned} x(t) &= a\cos(bt + c) \\ y(t) &= a\sin(bt + c). \end{aligned}$$

Again, this is motion parameterized on a circle.

Assuming that $a > 0$, we see that a must be the radius of the new circle. This satisfies the equation

$$x^2 + y^2 = a^2. \quad \text{[Try it!]}$$

We say that b is the *angular speed.* If $b = 1$, as in our original unit circle model, then it takes 2π seconds to complete one revolution on the unit circle.

Note what happens when we let $b = 2$.

$$\begin{aligned} x(t) &= \cos(2t) \\ y(t) &= \sin(2t) \end{aligned}$$

The period is cut in half! It takes only π seconds to complete one revolution.

Thus, one revolution can be completed in $P = \dfrac{2\pi}{b}$ seconds and the angular speed is b radians/sec.

The constant c is the initial angular displacement. It is related to the phase shift, which is

$$\phi = -\frac{c}{b}.$$

Thus, if we have the parametric equations

$$\begin{aligned} x(t) &= a\cos(bt + c) \\ y(t) &= a\sin(bt + c), \end{aligned}$$

our initial position is $(a\cos(c),\ a\sin(c))$ at $t = 0$, and we return to that position every $P = \dfrac{2\pi}{b}$ seconds.

The parametric curve is the circle of radius $R = a$, centered at the origin.

(#44) If $f(x) = \cos(x)$, we see that the derivatives of f occur in cycles of 4.

$$\begin{aligned} f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \\ f^{(4)}(x) &= \cos(x). \end{aligned}$$

So if n is a multiple of 4, then $f^{(n)}(x) = \cos(x)$.

If we divide by n by 4 and look at its remainder, we can easily find $f^{(n)}(x)$ from the table above.

(a) What is $f^{(77)}(x)$? We divide 77 by 4 and obtain 1 as the remainder. So

$$f^{(77)}(x) = f^{(1)}(x) = -\sin(x).$$

(b) What is $f^{(120)}(x)$? We see that 120 is a multiple of 4, so $f^{(120)}(x) = f(x) = \cos(x)$.

(#48) Use the basic limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

to find the limits.

(a) A reciprocal.

$$\lim_{t\to 0}\frac{2t}{\sin(t)} = 2\left(\lim_{t\to 0}\frac{t}{\sin(t)}\right) = 2\left(\lim_{t\to 0}\frac{1}{\left(\frac{\sin(t)}{t}\right)}\right) = 2\left(\frac{\lim_{t\to 0} 1}{\lim_{t\to 0}\frac{\sin(t)}{t}}\right) = 2\left(\frac{1}{1}\right) = 2.$$

(b) Direct substitution.

$$\lim_{x\to 0}\frac{\sin(x)}{5x+1} = \frac{0}{5(0)+1} = 0.$$

(c) A trick. We must mate each sine with an x.

$$\lim_{x\to 0}\frac{\sin(6x)}{\sin(5x)} = \lim_{x\to 0}\left(\left(\frac{\sin(6x)}{x}\right)\left(\frac{x}{\sin(5x)}\right)\right) = \left(\lim_{x\to 0}\frac{\sin(6x)}{x}\right)\left(\lim_{x\to 0}\frac{x}{\sin(5x)}\right).$$

Let's do each limit separately. Multiply the first one top-and-bottom by 6.

$$\lim_{x\to 0}\frac{\sin(6x)}{x} = \lim_{x\to 0}\frac{6\sin(6x)}{6x} = 6\left(\lim_{x\to 0}\frac{\sin(6x)}{6x}\right).$$

Now substitute $u = 6x$. As $x \to 0$, we have $u \to 0$.

$$\lim_{x\to 0}\frac{\sin(6x)}{x} = 6\left(\lim_{u\to 0}\frac{\sin(u)}{u}\right) = 6(1) = 6.$$

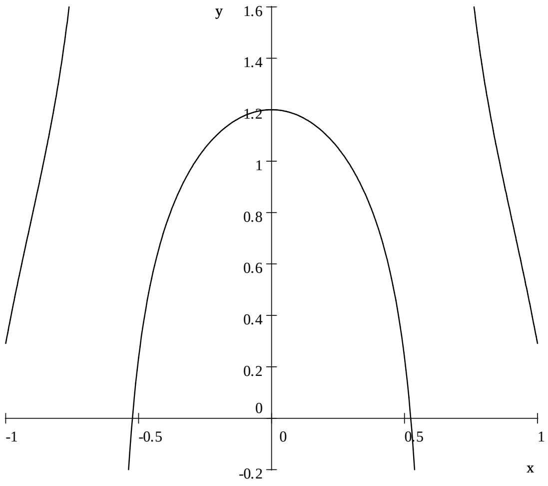
Similarly, we have

$$\lim_{x\to 0}\frac{x}{\sin(5x)} = \lim_{x\to 0}\frac{5x}{5\sin(5x)} = \frac{1}{5}\left(\lim_{x\to 0}\frac{5x}{\sin(5x)}\right) = \frac{1}{5}\left(\lim_{u\to 0}\frac{u}{\sin(u)}\right) = \frac{1}{5},$$

if we let $u = 5x$.

The product of the limits is $6\left(\frac{1}{5}\right) = \frac{6}{5}$.

We can verify this by a sketch of $f(x) = \frac{\sin(6x)}{\sin(5x)}$. The limit as $x \to 0$ should be $y = \frac{6}{5} = 1.2$.



It's perfect.

(d) Definition of tangent.

$$\lim_{x\to 0}\frac{\tan(2x)}{x}=\lim_{x\to 0}\frac{\left(\dfrac{\sin(2x)}{\cos(2x)}\right)}{x}=\lim_{x\to 0}\left(\frac{\sin(2x)}{x}\right)\left(\frac{1}{\cos(2x)}\right).$$

We should use the identity $\sin(2x)=2\sin(x)\cos(x)$ and we can directly substitute into the second limit.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(2x)}{x} &= \left(\lim_{x\to 0}\frac{2\sin(x)\cos(x)}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos(2x)}\right)\\ &= 2\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\left(\lim_{x\to 0}\cos(x)\right)\left(\frac{1}{\cos(2*0)}\right)\\ &= 2(1)(1)(1)=2.\end{aligned}$$

Section 2.7

(#4) Find the first-order derivatives.

Product Rule.

$$\begin{aligned}\left[e^{2x}\cos(4x)\right]' &= e^{2x}\left[\cos(4x)\right]'+\cos(4x)\left[e^{2x}\right]'\\ &= e^{2x}\left(-4\sin(4x)\right)+\cos(4x)\left(2e^{2x}\right)\\ &= -4e^{2x}\sin(4x)+2e^{2x}\cos(4x).\end{aligned}$$

It would be nice if you factored out the e^{2x}.

$$\left[e^{2x}\cos(4x)\right]'=e^{2x}\left(-4\sin(4x)+2\cos(4x)\right).$$

Why? If needed to find the second-order derivative, the Product Rule will be easily applied.

(#8) Rewrite $\left(\frac{1}{e}\right)^x=\frac{1}{e^x}=e^{-x}$.

$$\left[e^{-x}\right]'=-e^{-x}=-\frac{1}{e^x}.$$

We ARE allowed to have negative exponents in exponential functions when we write our final answers. Either answer is good.

(#10) The base is $b=4$.

$$\left[4^{-x^2}\right]'=4^{-x^2}\left(\ln(4)\right)\left[-x^2\right]'=4^{-x^2}\left(\ln(4)\right)(-2x).$$

I prefer to write all multiplicative constants in the front.

$$\left[4^{-x^2}\right]'=-2\ln(4)\,x\left(4^{-x^2}\right).$$

Polynomials come first, exponentials second, radicals third, trig. functions fourth, and finally natural log.

(#18) Product Rule.

$$\left[x^3\ln(x)\right]' = x^3\left[\ln(x)\right]' + \ln(x)\left[x^3\right]' = x^3\left(\frac{1}{x}\right) + \ln(x)\left(3x^2\right) = x^2 + 3x^2\ln(x).$$

(#20) Chain Rule.

$$\left[e^{\sin(2x)}\right]' = e^{\sin(2x)}\left[\sin(2x)\right]' = e^{\sin(2x)}\left(2\cos(2x)\right) = 2e^{\sin(2x)}\cos(2x).$$

(#22) Chain Rule twice.

$$\begin{aligned}\left[\ln\left(\sin\left(x^2\right)\right)\right]' &= \frac{1}{\sin(x^2)}\left[\sin\left(x^2\right)\right]' = \frac{1}{\sin(x^2)}\left(\cos\left(x^2\right)\left[x^2\right]'\right) \\ &= \frac{1}{\sin(x^2)}\left(2x\cos\left(x^2\right)\right) = 2x\left(\frac{\cos(x^2)}{\sin(x^2)}\right).\end{aligned}$$

This can also be written as $2x\cot\left(x^2\right)$.

(#25) Chain Rule and simplification.

$$\begin{aligned}\left[\ln(\sec(x) + \tan(x))\right]' &= \frac{\left[\sec(x) + \tan(x)\right]'}{\sec(x) + \tan(x)} = \frac{\tan(x)\sec(x) + \sec^2(x)}{\sec(x) + \tan(x)} \\ &= \frac{\sec(x)\left(\tan(x) + \sec(x)\right)}{\sec(x) + \tan(x)} = \sec(x).\end{aligned}$$

(#30) Find the equation of the tangent line to $y = 2^x$ at $x = 1$.

$$y' = \left[2^x\right]' = 2^x\ln(2) = \ln(2)\left(2^x\right).$$

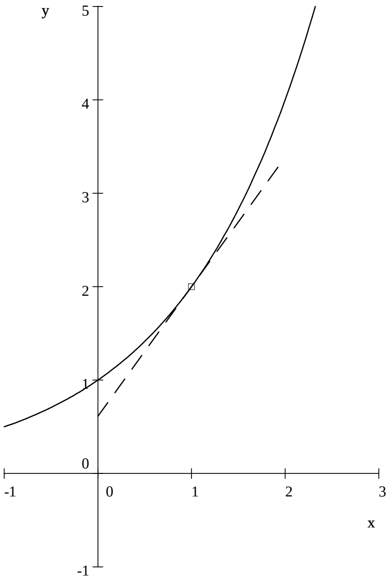
When $x = 1$, we have

$$f'(1) = \ln(2)\left(2^1\right) = 2\ln(2).$$

The point on the curve is $(1, 2)$, so the equation of the tangent line is

$$y = 2\ln(2)(x - 1) + 2.$$

Here's a sketch.



(#38) A bacterial population starts at 500 and doubles every four days. Find a formula for the population after t days.

Find the percentage rate of change in the population.

The population function must have this form:

$$P(t) = 500\left(2^{???}\right)$$

It "doubles" every 4 days, so the exponent must be $t/4$.

$$P(t) = 500\left(2^{t/4}\right).$$

When $t = 0$, we have $P(0) = 500$.✓

When $t = 4$, the exponent is 1, and we have a factor of 2, which doubles the population.

When $t = 8$, the exponent is 2, and we have doubled twice, etc.✓

The percent change is the *relative change* in the population:

$$\frac{P'(t)}{P(t)} = \frac{\left[500\left(2^{t/4}\right)\right]'}{500\left(2^{t/4}\right)} = \frac{500\left[\left(2^{t/4}\right)\right]'}{500\left(2^{t/4}\right)} = \frac{\left[\left(2^{t/4}\right)\right]'}{2^{t/4}} = \frac{2^{t/4}(\ln(2))\left[\frac{t}{4}\right]'}{2^{t/4}} = \frac{\ln(2)}{4} \doteq 0.173 \text{ or } +17.3\%.$$

This says that the population increases 17.3% (approx.) every day.

Also note that

$$[\ln(P(t))]' = \frac{P'(t)}{P(t)}.$$

This says that if $P(t)$ is an exponential function, we can always find the relative change by taking the natural log *first,* and then we take the derivative.

$$\ln\left(500 * 2^{t/4}\right) = \ln(500) + \ln\left(2^{t/4}\right) = \ln(500) + \frac{t}{4}\ln(2).$$

The derivative of a constant is zero.

$$\left[\ln\left(500 * 2^{t/4}\right)\right]' = \left[\ln(500) + \frac{t}{4}\ln(2)\right]' = 0 + \frac{1}{4}(\ln(2)) = \frac{\ln(2)}{4}.✓$$

(#64) The concentration of a certain chemical after t seconds of a reaction is given by

$$x(t) = \frac{10}{9e^{-10t} + 2}.$$

[NOTICE, by the way, that the horizontal axis is t and the vertical axis is x!

If we graphed this, we would suspect that there is a horizontal asymptote at $x = 5$.]

Show that $x'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 5.

$$\begin{aligned} x'(t) &= \left[\frac{10}{9e^{-10t} + 2}\right]' = 10\left[\left(9e^{-10t} + 2\right)^{-1}\right]' = 10\left(-1\left(9e^{-10t} + 2\right)^{-2}\left[9e^{-10t} + 2\right]'\right) \\ &= 10\left(-\frac{1}{\left(9e^{-10t} + 2\right)^2}\left(9\left(-10e^{-10t}\right)\right)\right) = \frac{900e^{-10t}}{\left(9e^{-10t} + 2\right)^2}. \end{aligned}$$

Remember that e to the anything is always positive. Thus, we see that $x'(t) > 0$ for all $t \geq 0$.

So the function is always increasing.

We find the limit as $t \to +\infty$.

$$\lim_{t\to+\infty} \frac{10}{9e^{-10t}+2} = \frac{\lim\limits_{t\to+\infty} 10}{\lim\limits_{t\to+\infty} \left(9e^{-10t}+2\right)} = \frac{10}{9\left(0\right)+2} = 5.$$

So the function approches $x = 5$, but never quite equals 5 because the denominator $\left(9e^{-10t}+2\right)$ is always greater than 2 for all $t \geq 0$.