Some Solutions to Assignment #04 – MATH 1401
Spring 2006

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Section 2.4

(#12) Find the first-order derivatives.

$$\left[\frac{6x-\frac{2}{x}}{x^2+\sqrt{x}}\right]' = \left[\frac{x\left(6x-\frac{2}{x}\right)}{x\left(x^2+\sqrt{x}\right)}\right]' = \left[\frac{6x^2-2}{x^3+x^{3/2}}\right]'.$$

I'd rather do the Quotient Rule on the last one.

$$\begin{aligned} f'(x) &= \frac{\left(x^3+x^{3/2}\right)\left[6x^2-2\right]' - \left(6x^2-2\right)\left[x^3+x^{3/2}\right]'}{\left(x^3+x^{3/2}\right)^2} \\ &= \frac{\left(x^3+x^{3/2}\right)(12x) - \left(6x^2-2\right)\left(3x^2+\frac{3}{2}x^{1/2}\right)}{\left(x^3+x^{3/2}\right)^2}. \end{aligned}$$

Rats. We still have a fraction in the numerator! We will cheat a little bit. Factor out (2) from $\left(6x^2-2\right)$ and then distribute it.

$$\begin{aligned} f'(x) &= \frac{\left(x^3+x^{3/2}\right)(12x) - \left(3x^2-1\right)(2)\left(3x^2+\frac{3}{2}x^{1/2}\right)}{\left(x^3+x^{3/2}\right)^2} \\ &= \frac{\left(x^3+x^{3/2}\right)(12x) - \left(3x^2-1\right)\left(6x^2+3x^{1/2}\right)}{\left(x^3+x^{3/2}\right)^2}. \end{aligned}$$

We must still expand all numerator items, but it is not as ugly as before...

$$\begin{aligned} f'(x) &= \frac{12x^4+12x^{5/2} - \left(18x^4+9x^{5/2}-6x^2-3\sqrt{x}\right)}{\left(x^3+x^{3/2}\right)^2} \\ &= \frac{-6x^4+3x^{5/2}+6x^2+3\sqrt{x}}{\left(x^3+x^{3/2}\right)^2}. \end{aligned}$$

(#26) The Extended Product Rule is

$$[fghk]' = f'ghk + fg'hk + fgh'k + fghk'.$$

$\left[\sqrt{x}\,(x-2)\,(x+1)\,(x+5)\right]' =$

$\left[x^{1/2}\right]'(x-2)\,(x+1)\,(x+5) + \sqrt{x}\,[x-2]'\,(x+1)\,(x+5) + \sqrt{x}\,(x-2)\,[x+1]'\,(x+5) +$
$\sqrt{x}\,(x-2)\,(x+1)\,[x+5]' =$

$\frac{1}{2}x^{-1/2}\,(x-2)\,(x+1)\,(x+5) + \sqrt{x}\,(x+1)\,(x+5) + \sqrt{x}\,(x-2)\,(x+5) + \sqrt{x}\,(x-2)\,(x+1) =$

$\dfrac{(x-2)\,(x+1)\,(x+5)}{2\sqrt{x}} + \sqrt{x}\,(x+1)\,(x+5) + \sqrt{x}\,(x-2)\,(x+5) + \sqrt{x}\,(x-2)\,(x+1)\,.$

If you wanted to be really spiffy, you could combine these with the LCD $= 2\sqrt{x}$.

$$\begin{aligned}
f'(x) &= \frac{(x-2)(x+1)(x+5)}{2\sqrt{x}} + \frac{2\sqrt{x}\left(\sqrt{x}\left((x+1)(x+5)+(x-2)(x+5)+(x-2)(x+1)\right)\right)}{2\sqrt{x}} \\
&= \frac{(x-2)(x+1)(x+5)}{2\sqrt{x}} + \frac{2x\left((x+1)(x+5)+(x-2)(x+5)+(x-2)(x+1)\right)}{2\sqrt{x}} \\
&= \frac{x^3+4x^2-7x-10+\left(6x^3+16x^2-14x\right)}{2\sqrt{x}} \\
&= \frac{7x^3+20x^2-21x-10}{2\sqrt{x}} = \frac{7}{2}x^{5/2}+10x^{3/2}-\frac{21}{2}x^{1/2}-\frac{5}{\sqrt{x}}.
\end{aligned}$$

If we had expanded the original:

$$\sqrt{x}(x-2)(x+1)(x+5) = x^{7/2}+4x^{5/2}-7x^{3/2}-10x^{1/2},$$

we could have used the Simple Power Rule four times!

I also asked you to evaluate $f'(4)$.

$$\begin{aligned}
f'(4) &= \frac{(4-2)(4+1)(4+5)}{2\sqrt{4}} + \sqrt{4}(4+1)(4+5) + \sqrt{4}(4-2)(4+5) + \sqrt{4}(4-2)(4+1) \\
&= \frac{(6)(5)(9)}{4} + (2)(5)(9) + (2)(2)(9) + (2)(2)(5) = \frac{427}{2} = 213.5.
\end{aligned}$$

(#38) If the baseball has mass M kg at speed 45 m/sec and the bat has mass 1.05 kg at speed 40 m/sec., then the ball's initial speed is

$$\mu(M) = \frac{86.625-45M}{M+1.05} \text{ m/sec.}$$

We have

$$\begin{aligned}
\frac{d\mu}{dM} &= \mu'(M) = \frac{(M+1.05)\,[86.625-45M]' - (86.625-45M)\,[M+1.05]'}{(M+1.05)^2} \\
&= \frac{(M+1.05)(-45)-(86.625-45M)(1)}{(M+1.05)^2} \\
&= -\frac{133.88}{(M+1.05)^2}. \quad \text{This is always negative for } M>0.
\end{aligned}$$

The derivative is the instantaneous rate of change in the ball's initial speed with respect to its mass.

It should make sense that if we throw a more massive ball, then the ball's initial speed *decreases.*

Section 2.5

(#6) Find all the first-order derivatives via some form of the Chain Rule.

General Power Rule.

$$\left[\left(x^3+4x\right)^{1/2}\right]' = \frac{1}{2}\left(x^3+4x\right)^{-1/2}\left[x^3+4x\right]' = \frac{3x^2+4}{2\sqrt{x^3+4x}}.$$

The Outer is $f(u) = u^{1/2}$ and $f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$.

The Inner is $g(x) = x^3+4x$ and $g'(x) = 3x^2+4$.

$$[f(g(x))]' = f'(g(x))\,g'(x) = \frac{1}{2\sqrt{(x^3+4x)}}\left(3x^2+4\right) = \frac{3x^2+4}{2\sqrt{x^3+4x}}.\checkmark$$

(#14) Multiplicative constant. General Power Rule.

$$\frac{1}{8}\left[\left(x^3+4\right)^5\right]' = \frac{1}{8}\left(5\left(x^3+4\right)^4\left[x^3+4\right]'\right) = \frac{1}{8}\left(5\left(x^3+4\right)^4\left(3x^2\right)\right) = \frac{15}{8}x^2\left(x^3+4\right)^4.$$

It's probably easier to expand the stuff inside the radical.

$$4x^2+\left(8-x^2\right)^2 = x^4-12x^2+64.$$

Thus, we have the General Power Rule again.

$$\begin{aligned}\left[\left(x^4-12x^2+64\right)^{1/2}\right]' &= \frac{1}{2}\left(x^4-12x^2+64\right)^{-1/2}\left[x^4-12x^2+64\right]' \\ &= \frac{4x^3-24x}{2\sqrt{x^4-12x^2+64}}.\end{aligned}$$

(#18) It's probably easier to expand the stuff inside the radical.

$$4x^2+\left(8-x^2\right)^2 = x^4-12x^2+64.$$

Thus, we have the General Power Rule again.

$$\begin{aligned}\left[\left(x^4-12x^2+64\right)^{1/2}\right]' &= \frac{1}{2}\left(x^4-12x^2+64\right)^{-1/2}\left[x^4-12x^2+64\right]' \\ &= \frac{4x^3-24x}{2\sqrt{x^4-12x^2+64}}.\end{aligned}$$

(#22) General Power Rule first, and then Product Rule?

$$\left[\left(\left(x^2+1\right)\left(\sqrt{x}+1\right)^3\right)^{1/2}\right]' = \frac{1}{2}\left(\left(x^2+1\right)\left(\sqrt{x}+1\right)^3\right)^{-1/2}\left[\left(x^2+1\right)\left(\sqrt{x}+1\right)^3\right]' =$$

$$\frac{1}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}\left(\left(x^2+1\right)\left[\left(\sqrt{x}+1\right)^3\right]' + \left(\sqrt{x}+1\right)^3\left[x^2+1\right]'\right) =$$

$$\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\left(\left(x^2+1\right)\left(3(\sqrt{x}+1)^2\left[x^{1/2}+1\right]'\right)+(\sqrt{x}+1)^3(2x)\right)=$$

$$\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\left(\left(x^2+1\right)\left(3(\sqrt{x}+1)^2\left(\frac{1}{2\sqrt{x}}\right)\right)+(\sqrt{x}+1)^3(2x)\right)=$$

$$\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\left(\frac{3\left(x^2+1\right)(\sqrt{x}+1)^2}{2\sqrt{x}}+(\sqrt{x}+1)^3(2x)\right).$$

We should combine the two terms in the parenthese. LCD $= 2\sqrt{x}$.

$$\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\left(\frac{3\left(x^2+1\right)(\sqrt{x}+1)^2}{2\sqrt{x}}+\frac{(\sqrt{x}+1)^3(2x)(2\sqrt{x})}{2\sqrt{x}}\right)=$$

$$\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\left(\frac{3\left(x^2+1\right)(\sqrt{x}+1)^2+4x\sqrt{x}(\sqrt{x}+1)^3}{2\sqrt{x}}\right)=$$

$$\frac{3\left(x^2+1\right)(\sqrt{x}+1)^2+4x\sqrt{x}(\sqrt{x}+1)^3}{4\sqrt{x}\sqrt{(x^2+1)(\sqrt{x}+1)^3}}.$$

We could factor out $(\sqrt{x}+1)^2$ from the numerator, and we would obtain the same result as I achieved when I approached it differently in my handout. [See handout.]

(#24) Find the equation of the tangent line to $f(x)=\frac{6}{x^2+4}$ when $x=a=-2$.

$$f'(x)=6\left[\left(x^2+4\right)^{-1}\right]'=6\left(-1\left(x^2+4\right)^{-2}\left[x^2+4\right]'\right)=-\frac{6}{(x^2+4)^2}(2x).$$

When $x=-2$, the slope of the tangent line is

$$f'(-2)=-\frac{12(-2)}{\left((-2)^2+4\right)^2}=\frac{3}{8}.$$

Since $f(-2)=\frac{6}{(-2)^2+4}=\frac{3}{4}$, we know the point of tangency is $\left(-2,\frac{3}{4}\right)$.

The equation of the tangent line is

$$y-\frac{3}{4}=\frac{3}{8}(x-(-2)) \quad\text{or}\quad y=\frac{3}{8}(x+2)+\frac{3}{4}.$$

(#26) If the position function is $s(t) = \dfrac{60t}{\sqrt{t^2+1}}$, find $v(2)$. Quotient Rule.

$$\begin{aligned}
v(t) &= s'(t) = 60\left[\frac{t}{\sqrt{t^2+1}}\right]' = 60\left(\frac{\sqrt{t^2+1}\,[t]' - t\left[\left(t^2+1\right)^{1/2}\right]'}{\left(\sqrt{t^2+1}\right)^2}\right) \\
&= 60\left(\frac{\sqrt{t^2+1} - t\left(\frac{1}{2}\left(t^2+1\right)^{-1/2}(2t)\right)}{t^2+1}\right) \\
&= 60\left(\frac{\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}\right) = 60\left(\frac{\frac{\sqrt{t^2+1}\sqrt{t^2+1}}{\sqrt{t^2+1}} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}\right) \\
&= 60\left(\frac{\left(\frac{t^2+1-t^2}{\sqrt{t^2+1}}\right)}{t^2+1}\right) = 60\left(\frac{\left(\frac{1}{\sqrt{t^2+1}}\right)}{t^2+1}\right) = 60\left(\frac{1}{\left(t^2+1\right)^{1/2}}\right)\left(\frac{1}{t^2+1}\right) \\
&= \frac{60}{\left(t^2+1\right)^{3/2}}.
\end{aligned}$$

The instantaneous velocity at $t = 2$ is $\dfrac{60}{\left(2^2+1\right)^{3/2}} = \dfrac{60}{5\sqrt{5}} = \dfrac{12\sqrt{5}}{5}$ units/sec.

(#38) I did this one in class.

(#42) If $f(x) = x^5 + 4x - 2$ and $g(x) = f^{-1}(x)$, then find $g'(-2)$.

We need to find x when $a = -2$ and

$$f(x) = a = -2 = x^5 + 4x - 2 \Rightarrow x^5 + 4x = 0 \Rightarrow x\left(x^4+4\right) = 0 \Rightarrow x = 0.$$

Thus, we have $g(-2) = x = 0$.

We have the Inverse Derivative formula.

$$\begin{aligned}
g'(a) &= \frac{1}{f'(g(a))} = \frac{1}{f'(0)}. \\
f'(x) &= 5x^4 + 4 \Rightarrow f'(0) = 4. \\
g'(-2) &= \frac{1}{4}.
\end{aligned}$$

Section 2.6

(#6) Find the first-order derivatives.

$$4\left[\sec\left(x^2\right)\right]' - 3\left[\cot(x)\right]' = 4$$

In the first term, we have

Outer: $f(u) = \sec(u)$, $f'(u) = \tan(u)\sec(u)$ and

Inner: $g(x) = x^2$, $g'(x) = 2x$.

$$4\tan\left(x^2\right)\sec\left(x^2\right)(2x) - 3\left(-\csc^2(x)\right) = 8x\tan\left(x^2\right)\sec\left(x^2\right) + 3\csc^2(x).$$

(#10) General Power Rule.

$$\begin{aligned}\left[\left(\sin^2(x)+2\right)^{1/2}\right]' &= \frac{1}{2}\left(\sin^2(x)+2\right)^{-1/2}\left[\sin^2(x)+2\right]' \\ &= \frac{1}{2\sqrt{\sin^2(x)+2}}\left(2\sin(x)\cos(x)\right) \\ &= \frac{\sin(x)\cos(x)}{\sqrt{\sin^2(x)+2}}.\end{aligned}$$

Remember that $\sin^2(x)$ is really $(\sin(x))^2$.

(#16) Product Rule.

$$\begin{aligned}\left[x^2\sec^2(3x)\right]' &= x^2\left[(\sec(3x))^2\right]' + \sec^2(3x)\left[x^2\right]' \\ &= x^2\left(2\left(\sec(3x)^1\left[\sec(3x)\right]'\right)\right) + \sec^2(3x)(2x) \\ &= 2x^2\sec(3x)\left(3\tan(3x)\sec(3x)\right) + 2x\sec^2(3x) \\ &= 6x^2\tan(3x)\sec^2(3x) + 2x\sec^2(3x).\end{aligned}$$

(#18) General Power Rule.

$$\begin{aligned}\left[(\cos(x))^2\right]' - \left[(\sin(x))^2\right]' &= 2(\cos(x))^1\left[\cos(x)\right]' - 2(\sin(x))^1\left[\sin(x)\right]' \\ &= 2\cos(x)(-\sin(x)) - 2\sin(x)\cos(x) \\ &= -4\sin(x)\cos(x).\end{aligned}$$

This is also equivalent to

$$\left[\cos(2x)\right]' = -2\sin(2x) = -2\left(2\sin(x)\cos(x)\right) = -4\sin(x)\cos(x).\checkmark$$