Some Solutions to Assignment #03 – MATH 1401
Spring 2006

Kawai

Section 2.2

(#4) We only need to write out the limit definition in equation (2.1). [The other one is equivalent, but usually takes more steps, so we deem it inefficient.]

Compute $f'(a)$, if $f(x) = \dfrac{3}{x+1}$ and $a = 2$. Our pattern of substitution is

$$f(\quad) = \frac{3}{(\quad)+1}.$$

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{3}{(2+h)+1} - \frac{3}{2+1}}{h} = \lim_{h\to 0} \frac{\frac{3}{h+3} - \frac{1}{1}}{h} = \lim_{h\to 0} \frac{\frac{3}{h+3} - \frac{h+3}{h+3}}{h} \\ &= \lim_{h\to 0} \frac{\left(\frac{-h}{h+3}\right)}{h} = \lim_{h\to 0} \left(\frac{-h}{h+3}\right)\left(\frac{1}{h}\right) = -\lim_{h\to 0} \left(\frac{1}{h+3}\right)\left(\frac{h}{h}\right) \\ &= -\lim_{h\to 0} \left(\frac{1}{h+3}\right) = -\frac{1}{3}. \end{aligned}$$

Since some of you already know the Chain Rule, we can check it:

$$\begin{aligned} \left[\frac{3}{x+1}\right]' &= 3\left[(x+1)^{-1}\right]' = 3\left(-1(x+1)^{-2}[x+1]'\right) = -3\left(\frac{1}{(x+1)^2}\right)(1) \\ &= -\frac{3}{(x+1)^2}. \end{aligned}$$

When we evaluate this at $x = 2$, we have

$$f'(2) = -\frac{3}{(2+1)^2} = -\frac{1}{3}.\checkmark$$

(#10) Compute $f'(a)$ if $f(x) = x^2 - 2x + 1$. Use the limit formula.

Our pattern of substitution is

$$f(\quad) = (\quad)^2 - 2(\quad) + 1.$$

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - 2(a+h) + 1 - \left(a^2 - 2a + 1\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(h^2 + 2ah + a^2\right) - 2a - 2h + 1 - a^2 + 2a - 1}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2ah - 2h}{h} = \lim_{h\to 0} \frac{h(h + 2a - 2)}{h} = \lim_{h\to 0} (h + 2a - 2) = 2a - 2. \end{aligned}$$

Thus, we have $f'(x) = 2x - 2 \Rightarrow \left[x^2 - 2x + 1\right]' = 2x - 2$.

(#50) Give the units for the derivative function.

(a) $c(t)$ represents the amount of a chemical present, in grams, at time t minutes.

The units of the first derivative $\dfrac{dc}{dt}$ will be (the instantaneous rate of change in the amount of chemical present with respect to time) grams per minute.

If it is positive, then the amount is increasing with respect to time. If it is negative, then the amount is decreasing with respect to time.

(b) $p(x)$ represents the mass, in kg, of the first x meters of a pipe.

The units of the first derivative $\dfrac{dp}{dx}$ will be (the instantaneous rate of change in mass with respect to linear distance) kg per meter.

This is density. When this value is large, then the wire is very dense. When x changes by a little, the increased mass would be large.

(#60) We want the limit below to resemble the derivative formula $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

We must choose $f(x)$ and a wisely. Here is the limit:

$$\lim_{h\to 0}\frac{(h-1)^2-1}{h} = ???$$

Since the only term which involves h is $(h-1)^2$, we can probably guess that

$$f(a+h)=(h-1)^2 \quad \text{and} \quad f(a)=1.$$

Since $f(a+h)=(h-1)^2$ gives us a squared function, we should try

$$f(x)=x^2.$$

This gives us

$$f(a+h)=(a+h)^2.$$

Can this equal $(h-1)^2$? Yes. Rewrite $(a+h)^2$ as $(h+a)^2$.

$$(h+a)^2=(h-1)^2 \Rightarrow a=-1.$$

So if we choose $f(x)=x^2$ and $a=-1$, then

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{(-1+h)^2-(-1)^2}{h}=\lim_{h\to 0}\frac{(h-1)^2-1}{h}.\checkmark$$

Note that since you already know that $\left[x^2\right]'=2x$, this limit must equal $2(-1)=-2$. You should check this yourself!

Section 2.3

(#8) $\left[5\sqrt{s}-4s^2+3\right]'=5\left[s^{1/2}\right]'-4\left[s^2\right]'+[3]'=5\left(\frac{1}{2}s^{-1/2}\right)-4\,(2s)+0=\frac{5}{2}s^{-1/2}-8s.$

Never leave negative exponents in the final answer.

$$\left(\frac{5}{2}\right)\left(\frac{1}{\sqrt{s}}\right)-8s.\checkmark$$

(#12) $\left[12x-x^2-\frac{3}{\sqrt{x}}\right]'=12-2x-3\left[x^{-1/2}\right]'=12-2x-3\left(-\frac{1}{2}x^{-3/2}\right)=12-2x+\frac{3}{2}\left(\frac{1}{x^{3/2}}\right).$

(#18) Expand.

$$\begin{aligned}(x+1)\left(3x^2-4\right) &= 3x^3+3x^2-4x-4.\\ \left[3x^3+3x^2-4x-4\right]' &= 9x^2+6x-4.\end{aligned}$$

(#20) Divide and Conquer.

$$\begin{aligned}\frac{4x^2-x+3}{\sqrt{x}} &= \frac{4x^2}{\sqrt{x}}-\frac{x}{\sqrt{x}}+\frac{3}{\sqrt{x}}=4\left(\frac{x^2}{x^{1/2}}\right)-\frac{x}{x^{1/2}}+3\left(\frac{1}{x^{1/2}}\right)\\ &= 4x^{3/2}-x^{1/2}+3x^{-1/2}.\end{aligned}$$

Now use the Power Rule three times.

$$\begin{aligned}\left[4x^{3/2}-x^{1/2}+3x^{-1/2}\right]' &= 4\left(\frac{3}{2}x^{1/2}\right)-\frac{1}{2}x^{-1/2}+3\left(-\frac{1}{2}x^{-3/2}\right)\\ &= 6x^{1/2}-\frac{1}{2}x^{-1/2}-\frac{3}{2}x^{-3/2}\\ &= 6\sqrt{x}-\frac{1}{2}\left(\frac{1}{\sqrt{x}}\right)-\frac{3}{2}\left(\frac{1}{x^{3/2}}\right).\end{aligned}$$

(#24) We must find the second order derivative, $\frac{d^2f}{dx^2}=f''(x)\,.$

$$f'(x)=\left[2x^4-3x^{-1/2}\right]'=8x^3+\frac{3}{2}x^{-3/2}.$$

We leave the negative exponents in this part, since we need the Simple Power Rule again.

$$f''(x)=\left[8x^3+\frac{3}{2}x^{-3/2}\right]'=24x^2+\frac{3}{2}\left(-\frac{3}{2}x^{-5/2}\right)=24x^2-\frac{9}{4}x^{-5/2}=24x^2-\frac{9}{4}\left(\frac{1}{x^{5/2}}\right).$$

(#32) If the position function is $s(t)=12t^3-6t-1$, then the velocity function is $v(t)=s'(t)\,.$

$$v(t)=\left[12t^3-6t-1\right]'=36t^2-6.$$

The acceleration function is $a(t)=v'(t)=s''(t)\,.$

$$a(t)=\left[36t^2-6\right]'=72t.$$

(#40) Find the equation of the tangent line to $y = f(x) = x^2 - 2x + 1$ at $x = 2$.

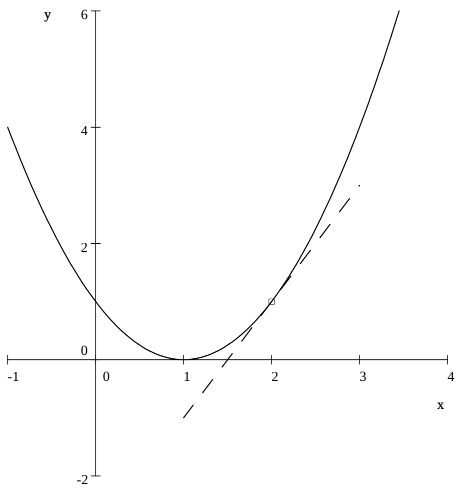
The slope of the tangent line will be $f'(2)$.

$$\begin{aligned} f'(x) &= \left[x^2 - 2x + 1\right]' = 2x - 2. \\ f'(2) &= 2(2) - 2 = 2. \end{aligned}$$

The point on the curve is $P(2, f(2)) = (2, 1)$, so the line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= 2(x - 2) \quad \text{or} \\ y &= 2(x - 2) + 1. \end{aligned}$$

Here's a sketch.



(#56) A rod of nonhomogeneous material extends from $x = 0$ to $x = 4$.

The mass of the portion of the rod from $x = 0$ to $x = t$ is given by $m(t) = 3t^2$ kg.

Compute $m'(t)$ and explain why it represents the *density* of the rod.

$$m'(t) = \left[3t^2\right]' = 6t.$$

The units of $m'(t)$ must be kilograms per unit distance (could be meters). In Physics problems, density is given as mass per length or mass per length^2 or mass per length^3. This density function tells us that the rod is much denser on the $x = 4$ end of the rod since we are acquiring more mass per length there. Hence, the rod is "lighter" (less dense) on the $x = 0$ end of the rod.

Section 2.4

(#2) Product Rule.

$$\begin{aligned}
f'(x) &= \left[\left(x^3-2x^2+5\right)\left(x^4-3x^2+2\right)\right]' \\
&= \left(x^3-2x^2+5\right)\left[\left(x^4-3x^2+2\right)\right]'+\left(x^4-3x^2+2\right)\left[\left(x^3-2x^2+5\right)\right]' \\
&= \left(x^3-2x^2+5\right)\left(4x^3-6x\right)+\left(x^4-3x^2+2\right)\left(3x^2-4x\right).
\end{aligned}$$

The only way to really check this is to multiply it out...

$7x^6-12x^5-15x^4+44x^3+6x^2-38x$.

In turns out, it's probably easier to expand the original, and then do the Simple Power Rule.

$$\left(x^3-2x^2+5\right)\left(x^4-3x^2+2\right)=x^7-2x^6-3x^5+11x^4+2x^3-19x^2+10.$$

You can easily verify that our first answer was correct!

(#10) Quotient Rule.

$$\begin{aligned}
\left[\frac{x^2+2x+5}{x^2-5x+1}\right]' &= \frac{\left(x^2-5x+1\right)\left[x^2+2x+5\right]'-\left(x^2+2x+5\right)\left[x^2-5x+1\right]'}{\left(x^2-5x+1\right)^2} \\
&= \frac{\left(x^2-5x+1\right)(2x+2)-\left(x^2+2x+5\right)(2x-5)}{\left(x^2-5x+1\right)^2} \\
&= \frac{\left(2x^3-8x^2-8x+2\right)-\left(2x^3-x^2-25\right)}{\left(x^2-5x+1\right)^2} \\
&= \frac{-7x^2-8x+27}{\left(x^2-5x+1\right)^2}.
\end{aligned}$$