## Class Log for MATH 1401-001 (Calculus I)

- Wednesday, 03/30:

  We actually didn't start the related rates (Section 4.1) last time. We'll finish them today.

- First, we want to revisit the differentials again.

  In our last episode, we used the formula for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ to obtain the differential equation:
  $$dV = 4\pi r^2 \, dr.$$

  If the value of $r$ was measured out to be $r = 3.000$, but with uncertainty $\pm 0.002$, then absolute (maximum) error is *approximated* by the differential $dV$:
  $$dV = 4\pi \left(3^2\right)(0.002) \doteq 0.22619 \text{ cubic units.}$$

  If we compare this error to the original volume (as calculated by the formula), then we have the *relative* error, which is usually expressed as a percentage.

  In this case, we have
  $$\frac{dV}{V} = \frac{0.22619}{\left(\frac{4}{3}\pi * 3^3\right)} = 0.002 = 0.2\%.$$

- In theory, we can calculate the value of $\frac{dV}{V}$ using the differentials:
  $$\frac{dV}{V} = \frac{4\pi r^2 \, dr}{\left(\frac{4}{3}\pi r^3\right)} = 3\left(\frac{dr}{r}\right).$$

  This is a fantastic result. Since the quantity $\frac{dr}{r}$ is the approximation for the *relative error* in the measurement of $r$, we see that the corresponding *relative error* in $V$ is three times that percentage.

  We said that the radius was $r = 3.000 \pm 0.002$. In terms of relative error, that's
  $$\frac{dr}{r} \approx \frac{0.002}{3.000} \doteq 0.06667\%$$

  According to the previous argument, we have
  $$\frac{dV}{V} = 3\left(\frac{dr}{r}\right) = 3\left(\frac{0.002}{3.000}\right) = 0.002 = 0.2\%.\checkmark$$

  So we see that the original procedure checks out okay!

- When we encounter related rates problems, we typically have some sort of geometric situation for which there is a corresponding (and hopefully nearly obvious) formula. We perform the differential step on this formula which yields a differential equation. We solve this equation for the appropriate element.

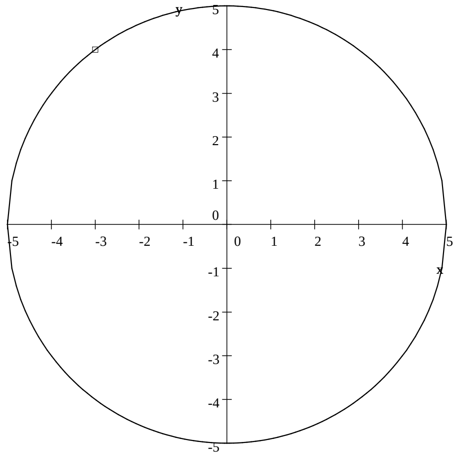
Example: We have a particle on a circular track.

$$x^2 + y^2 = 25$$

At some specific point in time, the particle is located at $(-3, 4)$.

At that instant, we measure the velocity in the x-axis direction, and find that $\dfrac{dx}{dt} = +7$ units/sec.

Consider the diagram:



The particle must be moving clockwise or counterclockwise along the circle.

Since $\dfrac{dx}{dt}$ is positive, we know that the x-coordinate of the particle is increasing.

Thus, from this current position, we know that the particle must be moving clockwise. So we expect the y-coordinate to increase,

and thus, $\dfrac{dy}{dt}$ should be positive also.

We can analyze the following mechanical step in two ways:

1. Notice that we can do the differential step to this equation.

$$\begin{aligned} x^2 + y^2 &= 25 \\ d\left[x^2\right] + d\left[y^2\right] &= d\left[25\right] \\ 2x\ dx + 2y\ dy &= 0. \end{aligned}$$

[This part is an added bonus...] Did you also notice that we could find the *implicit derivative* here also? We would only need to divide everything by $dx$, and we would have:

$$\begin{aligned} 2x\ \frac{dx}{dx} + 2y\ \frac{dy}{dx} &= 0 \\ 2x + 2y\left(y'\right) &= 0 \\ y' &= \frac{-2x}{2y} = -\frac{x}{y}.\checkmark \end{aligned}$$

What we really want to do is divide every term in

$$2x\ dx + 2y\ dy = 0$$

by $dt$! This will give us derivatives of $x$ and $y$ with respect to time.

$$2x\ \frac{dx}{dt} + 2y\ \frac{dy}{dt} = 0.$$

This is the related rates equation! We see that there are four variables: the current values of $x$ and $y$, and the two associated instantaneous rates of change.

1. Notice that we achieve the same goal by assuming that there are $t$'s hidden inside the $x$'s and $y$'s.

   If that is true, then we can use the Chain Rule and take the derivative of the original equation with respect to $t$.

$$\begin{aligned} \frac{d}{dt}\left[x^2\right] + \frac{d}{dt}\left[y^2\right] &= \frac{d}{dt}[25] \\ 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} &= 0. \end{aligned}$$

   This method is certainly more direct, but it doesn't give us as many options as in the first method. You may use either method.

- Now we can answer the question. If our current position is $(-3, 4)$ and the instantaneous rate of change is $\frac{dx}{dt} = +7$, then what is the instantaneous value of $\frac{dy}{dt}$?

$$\begin{aligned} 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} &= 0 \\ 2(-3)(7) + 2(4)\left(\frac{dy}{dt}\right) &= 0 \\ 8\left(\frac{dy}{dt}\right) &= 42 \Rightarrow \frac{dy}{dt} = +\frac{21}{4} \text{ units/sec.} \end{aligned}$$

  Yes, the rate of change was positive, so it matched the diagram. Also, by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{21}{4}\right)}{7} = \frac{3}{4}.$$

  We note that this matches the implicit derivative formula:

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{(-3)}{4} = \frac{3}{4}.\checkmark$$

- Let's think about the formula for a sphere again. Suppose we are inflating a balloon, so that $\frac{dV}{dt}$ is a constant value, say $+4$ units$^3$/sec.

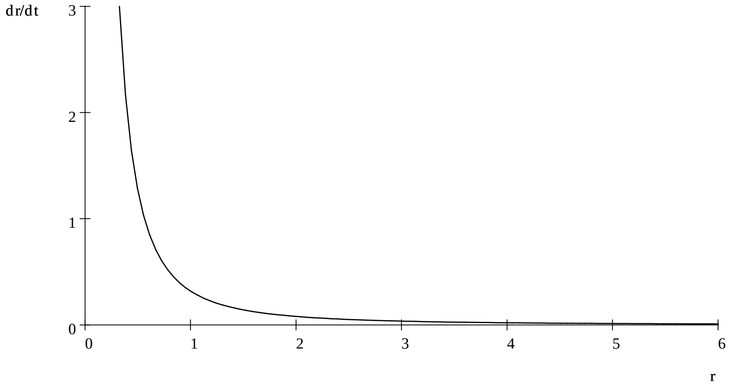
  At what does radius change? [What does this depend on?]

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{4}{3}\pi * \frac{d}{dt}\left[r^3\right] \\ \frac{dV}{dt} &= \frac{4}{3}\pi\left(3r^2\,\frac{dr}{dt}\right) = 4\pi r^2\left(\frac{dr}{dt}\right). \end{aligned}$$

  This says that the instantaneous rate of change in the radius with respect to time is equal to the change in volume per time, divided by the current value of the surface area.

$$\frac{dr}{dt} = \left(\frac{1}{4\pi r^2}\right)\left(\frac{dV}{dt}\right).$$

Since $\dfrac{dV}{dt}$ is constant, we can make a graph of $\dfrac{dr}{dt}$ vs. $r$. It will an inverse square graph!



- So if the current value of $r = 3$ units, what is the value of $\dfrac{dr}{dt}$?

$$\begin{aligned}
\frac{dV}{dt} &= 4\pi r^2 \left(\frac{dr}{dt}\right) \\
4 &= 4\pi \left(3^2\right) \left(\frac{dr}{dt}\right) \\
\frac{dr}{dt} &= \frac{4}{36\pi} \doteq +0.035368 \text{ units/sec.}
\end{aligned}$$