

## Class Log for MATH 1401-001 (Calculus I)

- Wednesday, 03/02:

We introduce the concept of Implicit Differentiation [Section 3.6].

- Previously, we looked at this circle:  $x^2 + y^2 = 4$ .

We separated it into two explicit functions:

$$f_1(x) = -\sqrt{4-x^2} \quad \text{and} \quad f_2(x) = \sqrt{4-x^2}.$$

So we can clearly find the derivative at  $(-\sqrt{2}, -\sqrt{2})$  as long as we can decide on *which* function our point lies.

In this case, we know that it lies on the bottom half of the circle:  $y = -(4-x^2)^{1/2}$ .

We apply the Chain Rule.

$$-\left[(4-x^2)^{1/2}\right]' = -\left(\frac{1}{2}(4-x^2)^{-1/2}[4-x^2]'\right) = -\left(\frac{1}{2\sqrt{4-x^2}}(-2x)\right) = \frac{x}{\sqrt{4-x^2}}.$$

When  $x = -\sqrt{2}$ , we have  $y' = \frac{-\sqrt{2}}{\sqrt{4-(\sqrt{2})^2}} = -\frac{\sqrt{2}}{\sqrt{2}} = -1$ .

This matches our previous answer using parametric equations.

- The Chain Rule offers yet another option.

If we have an implicit form such as  $x^2 + y^2 = 4$ , and we assume that there  $x$ 's HIDDEN INSIDE of the  $y$ 's, then we can perform *implicit differentiation*.

So even if it is impossible to derive an explicit function from the implicit equation, we can still assume that  $y$  is really  $f(x)$ .

This gives us the following implicit derivatives:

$$\frac{d}{dx}[y] = y'.$$

This the slope of the tangent line at a particular point. We will want to solve for  $y'$  in the end.

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[(f(x))^2] = 2f(x) * f'(x) = 2yy'.$$

This is an example of the General Power Rule. Similarly, we have

$$\frac{d}{dx}[y^n] = ny^{n-1}y'.$$

We also have the other forms of the Chain Rule.

$$\frac{d}{dx}[\sin(y)] = \cos(y) * y'$$

$$\frac{d}{dx}[e^y] = e^y * y'.$$

- So find the derivative with respect to  $x$  on both sides of the equality.

$$\frac{d}{dx} [x^2] + \frac{d}{dx} [y^2] = \frac{d}{dx} [4]$$

$$2x + 2yy' = 0.$$

Solve for  $y'$ .

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = -\frac{x}{y}.$$

This tells us that the slope of the tangent line depends on *both*  $x$  and  $y$ . So we no longer need to know on which half of the circle our point lies. We merely substitute in the coordinates of our point of interest.

For our point  $(-\sqrt{2}, -\sqrt{2})$ , we obtain the value  $y' = -1$ . ✓

- Note that if  $y = \sqrt{4 - x^2}$ , we have  $y' = -\frac{x}{y} = -\frac{x}{\sqrt{4 - x^2}}$  which is the explicit derivative.

If  $y = -\sqrt{4 - x^2}$ , we have  $y' = -\frac{x}{y} = -\frac{x}{(-\sqrt{4 - x^2})} = \frac{x}{\sqrt{4 - x^2}}$ , which, again, is the explicit derivative.

Also note that if we had used the parametric equations

$$\begin{aligned} x &= 2 \cos(t) \\ y &= 2 \sin(t), \end{aligned}$$

we obtain  $y' = -\frac{x}{y} = -\frac{2 \cos(t)}{2 \sin(t)} = -\cot(t)$ . ✓

- So all of the techniques fit together!

Let's answer that leftover question from last time. Find the implicit derivative  $\frac{dy}{dx}$  on the curve

$$y^5 + \sin(y) = x^2 + x, \text{ at } (0, 0).$$

$$\frac{d}{dx} [y^5] + \frac{d}{dx} [\sin(y)] = \frac{d}{dx} [x^2 + x]$$

$$5y^4 y' + \cos(y) * y' = 2x + 1$$

$$(5y^4 + \cos(y)) y' = 2x + 1$$

$$y' = \frac{2x + 1}{5y^4 + \cos(y)}.$$

When  $x = 0$  and  $y = 0$ , we have  $y' = 1$ . The slope of the tangent line should be 1.

- Now we need to look back at a related problem – the inverse function problem.

We recall that finding the inverse algebraically is, in theory, quite easy. We merely exchange the roles of  $y$  and  $x$  in our equation, and then we have it.

Thus, if  $y = f(x) = 2x + 7$ , we can find  $f^{-1}(x)$ .

We exchange  $y$  for  $x$  and then solve for  $y$ .

$$\begin{aligned} x &= 2y + 7 \\ x - 7 &= 2y \end{aligned}$$

$$y = \frac{x - 7}{2} = f^{-1}(x).$$

The inverse function has the following property under functional composition:

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(x)) = x.$$

Let's verify this with our example above.

$$f^{-1}(f(x)) = f^{-1}(2x + 7) = \frac{(2x + 7) - 7}{2} = \frac{2x}{2} = x. \checkmark$$

You should verify that  $f(f^{-1}(x)) = x$ .

- So if  $g(x)$  is the inverse of  $f(x)$ , then we have

$$f(g(x)) = x$$

and then we can apply the Chain Rule!

$$\begin{aligned} [f(g(x))]' &= [x]' \\ f'(g(x)) * g'(x) &= 1 \end{aligned}$$

$$g'(x) = \frac{1}{f'(g(x))}.$$

This the Derivative of the Inverse formula.

- We can determine  $[\ln(x)]'$ .

If  $f(x) = e^x$ , then its inverse is  $g(x) = \ln(x)$ .

Since  $f'(x) = e^x$ , we have

$$g'(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Thus, we have  $[g(x)]' = [\ln(x)]' = \frac{1}{x}$ .

- This also gives us the Logarithmic form of the Chain Rule:

$$\frac{d}{dx} [\ln(u)] = \frac{1}{u} * u' = \frac{u'}{u}.$$