Class Log for MATH 1401-001 (Calculus I)

- Monday, 02/28:

  We reviewed some concepts from Parametric Curves/Equations [Section 1.7]. Some of the students had not seen this before.

  For 2-D Parametric Curves, we have two function, $x(t)$ and $y(t)$. Each is clearly a function of t. They share the same domain, some subset of the real numbers for t (t is the parameter).

- A well-known example...

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

So the domain is $[0, 2\pi]$.

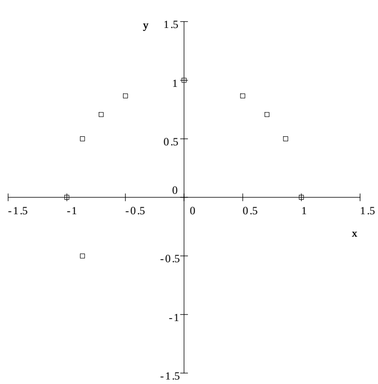
For each value of t in the domain, we obtain a point $(x(t), y(t))$ which lies on the parametric curve.

If $x(t)$ and $y(t)$ are both continuous, then the parametric curve is continuous also.

We choose some obvious angles (values for t) and calculate some coordinates...

t	$x(t)$	$y(t)$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1

t	$x(t)$	$y(t)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
π	-1	0
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$



It should be clear to everyone that we are tracing out a unit circle. Since the domain is $[0, 2\pi]$, we see that the curve starts at $(1, 0)$, goes around counterclockwise, and then ends at $(1, 0)$.

So this parametric curve is a simple closed curve.

Sketch in the circle and then show the orientation by sketching in arrows on the circle in the counterclockwise direction. Be sure to note that $(1, 0)$ is both the starting and ending point!

- How can we show that the system of parametric equations is equivalent to the graph of the circle?

  Eliminate t.

$$\left\{ \begin{array}{c} x = \cos(t) \\ y = \sin(t) \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} x^2 = \cos^2(t) \\ y^2 = \sin^2(t) \end{array} \right\}$$

  If we add the equations to each other, we obtain $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$.

  We know that $x^2 + y^2 = 1$ is the equation of the unit circle.

- We can modify this set of equations to produce other parametric curves...

  1. If we want to change its radius, we have

$$\begin{aligned} x(t) &= 4\cos(t) \\ y(t) &= 4\sin(t) \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

     This circle has radius 4.

  2. Suppose we want to translate this circle so that its center is located at $(3, -2)$.

$$\begin{aligned} x(t) &= 3 + 4\cos(t) \\ y(t) &= -2 + 4\sin(t) \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

     We note this curve starts at $(7, -2)$ and again proceeds counterclockwise until it returns to $(7, -2)$ when $t = 2\pi$.

  3. Note what happens if we have this set of equations:

$$\begin{aligned} x(t) &= \sin(t) \\ y(t) &= \cos(t) \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

     We still have the unit circle, but we start at $(0, 1)$ and proceed *clockwise*. Again, we return to $(0, 1)$ when $t = 2\pi$.

  4. We can stretch the circle out a bit using these equations:

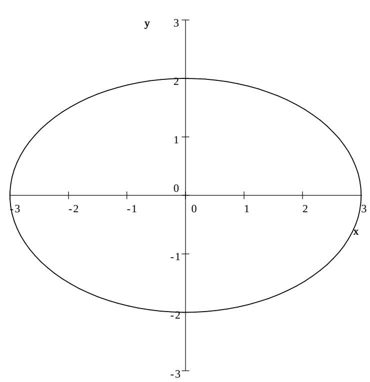
$$\begin{aligned} x(t) &= 3\cos(t) \\ y(t) &= 2\sin(t) \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

     This gives us $\frac{x}{3} = \cos(t)$ and $\frac{y}{2} = \sin(t)$.

     If we square both sides and then add them together, we obtain

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

     This is an ellipse. The orientation is counterclockwise.

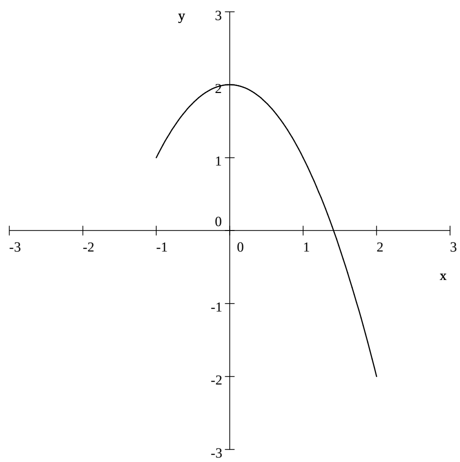


This is an ellipse.
The starting and ending point is $(3, 0)$ and the orientation is counterclockwise.

5. How do we parameterize function curves? Easy.
   If we have $y = f(x) = -x^2 + 2$, then let $x = t$.

$$\begin{aligned} x(t) &= t \\ y(t) &= -t^2 + 2 \qquad for \ -1 \leq t \leq 2. \end{aligned}$$

   This will give us a piece of the parabola $y = -x^2+2$. Do you see how we are substituting $x = t$?
   The starting point will be at $(-1, f(-1)) = (-1, 1)$ and the ending point will be at $(2, f(2)) = (2, -2)$.



6. We can reverse the orientation of this parameterization by simple algebra.
   Substitute $(-t)$ for t everywhere, including the domain statement.

$$\begin{aligned} x(t) &= -t \\ y(t) &= -(-t)^2 + 2 \qquad for \ -1 \leq -t \leq 2. \end{aligned}$$

   Since $-1 \leq -t \leq 2$, we can multiply through by (-1) and reverse the inequality signs. This gives us $-2 \leq t \leq 1$. We simplify the parameterization.

$$\begin{aligned} x(t) &= -t \\ y(t) &= -t^2 + 2 \qquad for \ -2 \leq -t \leq 1. \end{aligned}$$

   Check it! Does it begin at $(2, -2)$ and end at $(-1, 1)$? YES!

- Remember that the Leibniz form of the Chain Rule looks like this:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

  Again, this assumes that $y = f(u)$, the outer function, and $u = g(x)$, the inner function, are both differentiable.

  The final answer must be an expression in x, so really, we must remember that $\frac{dy}{du}$ must be evaluate at $u = g(x)$.

We can use the evaluation bar or the square brackets to reflect this:

$$\frac{dy}{dx} = \left(\left. \frac{dy}{du} \right|_{u=g(x)} \right) * \frac{du}{dx} = \left[\frac{dy}{du} \right]_{u=g(x)} * \frac{du}{dx}.$$

- Here is the parametric form of the Chain Rule. If $x = x(t)$ and $y = y(t)$, and both functions are differentiable with respect to t, then we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Leibniz said that derivatives acted like fractions. This is equivalent to

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

The derivative $\frac{dy}{dx}$ represents the slope of the tangent line to the parametric curve.

Often, we *cannot* eliminate the parameter t easily when we have parametric equations because $x(t)$ and $y(t)$ are very unwieldy.

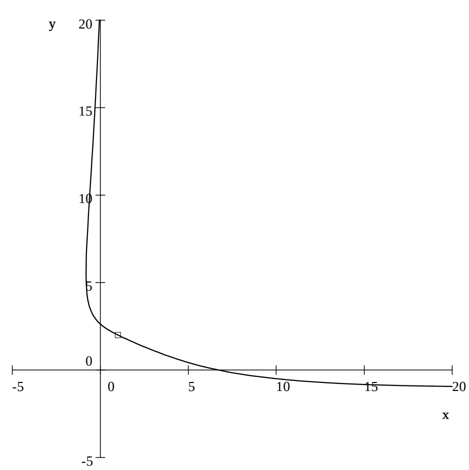
Example:

$$\begin{aligned} x &= e^t + \sin(t) \\ y &= e^{-t} + \cos(t) \end{aligned}$$

It turns out that there is NO way to eliminate t, and yet, it is easy to find $\frac{dy}{dx}$ for any value of t.

$$\begin{aligned} \frac{dy}{dt} &= -e^{-t} - \sin(t) \\ \frac{dx}{dt} &= e^t + \cos(t) \\ \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t} - \sin(t)}{e^t + \cos(t)}. \end{aligned}$$

Here's the parametric curve for $-\pi \leq t \leq \pi$:



When $t = 0$, we're located at $(1, 2)$. On the graph, this is denoted by the box.

The slope of the tangent line there is

$$\frac{dy}{dx} = \frac{-e^{-0} - \sin(0)}{e^0 + \cos(0)} = -\frac{1}{2}.$$

- Let's consider one of our circles again: $x^2 + y^2 = 4$.

$$\begin{aligned} x(t) &= 2\cos(t) \\ y(t) &= 2\sin(t), \qquad for\ 0 \le t \le 2\pi. \end{aligned}$$

  By our Chain Rule, we have

$$\frac{dx}{dt} = -2\sin(t) \quad \text{and} \quad \frac{dy}{dt} = 2\cos(t)$$

$$\frac{dy}{dx} = \frac{2\cos(t)}{-2\sin(t)} = -\cot(t).$$

  So if we know the central angle associated with a point on the circle, we can predict the slope of the tangent line.

  For example, in Quadrant III, if we choose $t = \dfrac{5\pi}{4}$, and land at the point $\left(-\sqrt{2}, -\sqrt{2}\right)$, we know that the slope of the tangent line must be $-\cot\left(\dfrac{5\pi}{4}\right) = -1$.

- We know that the circle mentioned above is NOT the graph of a function. It does not pass the Vertical Line Test.

  However, we know that we could solve $x^2 + y^2 = 4$ for y, and obtain:

$$\begin{aligned} y^2 &= 4 - x^2 \\ y &= \pm\sqrt{4 - x^2}. \end{aligned}$$

  This allows us to break up the circle into two parts

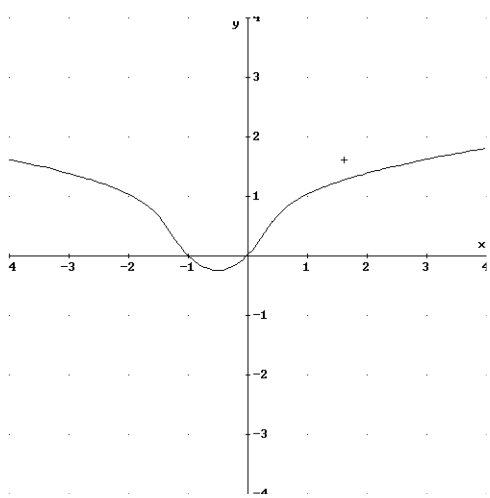
$$f_1(x) = -\sqrt{4 - x^2} \quad \text{and} \quad f_2(x) = \sqrt{4 - x^2}$$

  which are both functions.

  Thus, we say that the equation $x^2 + y^2 = 4$ is really a union of *implicit functions.* If we can solve for them *explicitly*, then we can write them down as regular functions. The algebra was not too horrible for this example. We were able to write down the two implicit functions explicitly.

- Here's a weird curve.

$$y^5 + \sin(y) = x^2 + x$$



It turns out that this curve *will* pass the Vertical Line Test! However, there is NO way to write out y as a function of x explicitly.

Worse, we can't define this curve with parametric equations either.

So how will we find $\dfrac{dy}{dx}$ at $(0, 0)$?