

## Class Log for MATH 1401-001 (Calculus I)

- Wednesday, 02/23:

More Chain Rule! [Section 3.5]

We've already covered the Chain Rules associated with the trig. functions.

We do have rules for  $[\csc(u)]'$  and  $[\cot(u)]'$  and the text neglects to point out some convenient identities.

For example, did you know this?

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x).$$

We often refer to this as a “Complementary Identity” because it relates the two angles,  $x$  and  $\left(\frac{\pi}{2} - x\right)$  which are complements of each other (their sum adds up to  $90^\circ$ ). We often say that “co” in “**co**sine” refers to the fact that the cosine actually calculates the sine of the **complementary** angle.

Anyway, we have this nice identity, and now we do some algebraic magic...

$$\frac{1}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{1}{\sin(x)} = \csc(x) \quad \text{and} \quad \frac{1}{\cos\left(\frac{\pi}{2} - x\right)} = \sec\left(\frac{\pi}{2} - x\right)$$

$$\csc(x) = \sec\left(\frac{\pi}{2} - x\right)$$

$$\begin{aligned} [\csc(x)]' &= \left[\sec\left(\frac{\pi}{2} - x\right)\right]' = \tan\left(\frac{\pi}{2} - x\right) \sec\left(\frac{\pi}{2} - x\right) \left[\frac{\pi}{2} - x\right]' \\ &= \cot(x) \csc(x) * (-1) \\ &= -\cot(x) \csc(x). \end{aligned}$$

Tangent and cotangent are complementary, as well as secant and cosecant.

- The General Power Rule

The outer function is  $f(u) = u^n$ . The inner function is  $u = g(x)$ .

$$[(g(x))^n]' = ???$$

The derivative of the outer function is  $f'(u) = n(u)^{n-1}$ .

We copy in  $g(x)$  and then multiply the result by  $g'(x)$ .

$$[f(g(x))]' = [(g(x))^n]' = n(g(x))^{n-1} * g'(x).$$

Thus, if the  $x$ 's are hidden inside of the  $u$ , then we have

$$\frac{d}{dx} [u^n] = nu^{n-1}u'.$$

Verify these examples:

1.  $\left[(x^3 + \sqrt{x})^9\right]' = ???$

The outer function is  $f(u) = u^9$ . The inner function is  $g(x) = x^3 + x^{1/2}$ .

$$\left[(x^3 + \sqrt{x})^9\right]' = 9(x^3 + \sqrt{x})^8 * [x^3 + x^{1/2}]' = 9(x^3 + \sqrt{x})^8 \left(3x^2 + \frac{1}{2}x^{-1/2}\right).$$

We would probably want to write that last factor in the form of a fraction. Eventually, we would want to set the entire derivative equal to zero and then solve.

$$9(x^3 + \sqrt{x})^8 \left(3x^2 + \frac{1}{2\sqrt{x}}\right) = 9(x^3 + \sqrt{x})^8 \left(\frac{3x^2(2\sqrt{x})}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}\right) =$$

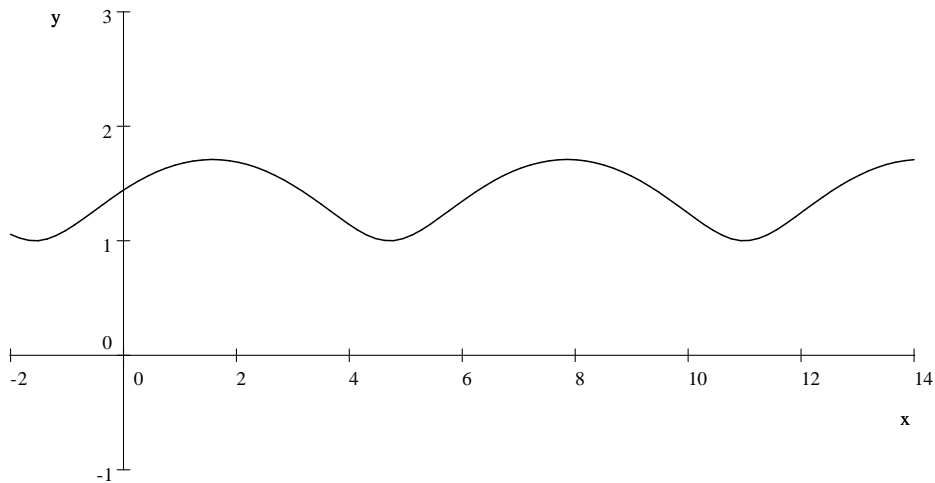
$$9(x^3 + \sqrt{x})^8 \left(\frac{6x^{5/2} + 1}{2\sqrt{x}}\right).$$

It turns out that the only real solution is  $x = 0$ , and, of course, this is unacceptable since this would give us a zero in the denominator.

2.  $\left[\sqrt[3]{2\sin(x) + 3}\right]' = \left[(2\sin(x) + 3)^{1/3}\right]' = \frac{1}{3}(2\sin(x) + 3)^{-2/3} [2\sin(x) + 3]' =$

$$\frac{1}{3}(2\sin(x) + 3)^{-2/3} (2\cos(x)) = \frac{2\cos(x)}{3(2\sin(x) + 3)^{2/3}}.$$

We believe that there is a horizontal tangent line at  $x = \frac{\pi}{2}, \frac{3\pi}{2}$ , etc. It's true!



3. General Power Rule twice!

$$\left[ \sqrt{1 + \sqrt{1 + \sqrt{x}}} \right]' = \left[ \left( 1 + (1 + x^{1/2})^{1/2} \right)^{1/2} \right]' = ???$$

The outer function is  $f(u) = x^{1/2}$ . The inner function is  $g(x) = 1 + (1 + x^{1/2})^{1/2}$ .

$$\left[ \left( 1 + (1 + x^{1/2})^{1/2} \right)^{1/2} \right]' = \frac{1}{2} \left( 1 + (1 + x^{1/2})^{1/2} \right)^{-1/2} * \underbrace{\left[ 1 + (1 + x^{1/2})^{1/2} \right]'}_{DO IT AGAIN!}$$

Same rule! The outer function is  $f(u) = u^{1/2}$ . The inner function is  $g(x) = 1 + x^{1/2}$ .

$$\left[ 1 + (1 + x^{1/2})^{1/2} \right]' = 0 + \frac{1}{2} (1 + x^{1/2})^{-1/2} * [1 + x^{1/2}]' = \frac{1}{2\sqrt{1 + \sqrt{x}}} * \left( 0 + \frac{1}{2} x^{-1/2} \right) =$$

$$\frac{\left( \frac{1}{2\sqrt{x}} \right)}{2\sqrt{1 + \sqrt{x}}} = \frac{1}{4\sqrt{x}\sqrt{1 + \sqrt{x}}}. \quad \text{Substitute this back into our original derivative.}$$

$$\begin{aligned} \left[ \left( 1 + (1 + x^{1/2})^{1/2} \right)^{1/2} \right]' &= \frac{1}{2\sqrt{1 + (1 + x^{1/2})^{1/2}}} \left( \frac{1}{4\sqrt{x}\sqrt{1 + \sqrt{x}}} \right) = \\ \frac{1}{2\sqrt{1 + \sqrt{1 + \sqrt{x}}}} \left( \frac{1}{4\sqrt{x}\sqrt{1 + \sqrt{x}}} \right) &= \frac{1}{8\sqrt{x}\sqrt{1 + \sqrt{x}}\sqrt{1 + \sqrt{1 + \sqrt{x}}}}. \end{aligned}$$

This never equals zero. (Thank goodness!)

• Chain Rule for Exponential Functions

$$\left[ e^{g(x)} \right]' = ???$$

The outer function is  $f(u) = e^u$  and the inner function is  $u = g(x)$ .

The derivative of the outer function is  $f'(u) = e^u$ .

We copy in  $g(x)$  and then multiply the result by  $g'(x)$ .

$$[f(g(x))] = \left[ e^{g(x)} \right]' = e^{g(x)} * g'(x).$$

Thus, if the  $x$ 's are hidden inside of the  $u$ , then we have

$$\frac{d}{dx} [e^u] = e^u u'.$$

Verify these examples:

1.  $\left[ e^{-x^2} \right]' = ???$

The outer function is  $f(u) = e^u$  and the inner function is  $g(x) = -x^2$ .

$$\left[ e^{-x^2} \right]' = e^{-x^2} * [-x^2]' = e^{-x^2} * (-2x) = -2xe^{-x^2}.$$

2.  $\left[ e^{\cos(x/3)} \right]' = e^{\cos(x/3)} * \left[ \cos \left( \frac{x}{3} \right) \right]' = e^{\cos(x/3)} * \left( -\sin \left( \frac{x}{3} \right) \right) * \left[ \frac{x}{3} \right]' =$

$$e^{\cos(x/3)} * \left( -\sin \left( \frac{x}{3} \right) \right) * \left( \frac{1}{3} \right) = -\frac{1}{3} e^{\cos(x/3)} \sin \left( \frac{x}{3} \right).$$

- We need an extension for the Exponential form of the Chain Rule.

The formula works okay when  $e$  is the base. What if we have a different, positive real number for the base.

$$[2^x]' = ???$$

We must rewrite “2” as  $e^{\ln(2)}$ . Now we have

$$2^x = \left(e^{\ln(2)}\right)^x = e^{\ln(2)*x}.$$

We can easily find this derivative.

$$\left[e^{\ln(2)*x}\right]' = e^{\ln(2)*x} * [\ln(2) * x]' = e^{\ln(2)*x} * \ln(2) = 2^x * \ln(2).$$

Remember that  $\ln(2)$  is only a multiplicative constant here!

Thus, if  $a > 0$  is our base, then we have

$$\frac{d}{dx} \left[ a^{g(x)} \right] = a^{g(x)} * \ln(a) * g'(x).$$

If the  $x$ 's are hidden inside of the  $u$ 's, then we have

$$[a^u]' = a^u * \ln(a) * u'.$$

Verify these examples:

1.  $[7^{\tan(2\theta)}]' = 7^{\tan(2\theta)} * \ln(7) * [\tan(2\theta)]' = 7^{\tan(2\theta)} * \ln(7) * \sec^2(2\theta) * [2\theta]' = 7^{\tan(2\theta)} * \ln(7) * \sec^2(2\theta) * 2 = 2 \ln(7) * 7^{\tan(2\theta)} * \sec^2(2\theta).$
4.  $\left[5^{\frac{x}{x^2+1}}\right]' = 5^{\frac{x}{x^2+1}} * \ln(5) * \left[\frac{x}{x^2+1}\right]' = 5^{\frac{x}{x^2+1}} * \ln(5) * \frac{(x^2+1)[x]' - x[x^2+1]'}{(x^2+1)^2} = 5^{\frac{x}{x^2+1}} * \ln(5) * \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} * \ln(5) * 5^{\frac{x}{x^2+1}}.$

The graph of  $y = 5^{\frac{x}{x^2+1}}$  should have horizontal tangents at  $x = \pm 1$ . It's true.

