Class Log for MATH 1401-001 (Calculus I)

- Monday, 02/07:

  We finished up some material from Section 2.10 and worked on Section 3.1.

- The most important things to remember about differentiability:

  1. A function is differentiable at $x = a$ if and only if $f'(a)$ exists.
     We already know that f must be continuous there, else the limit
     $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$
     cannot exist.
  2. A function is *smooth* if it has a tangent line, but, of course, if that tangent line is vertical, then the slope is undefined, and then it would not be differentiable there.
     So $f(x) = \sqrt[3]{x} = x^{1/3}$ is continuous everywhere, smooth everywhere, but not differentiable at $x = 0$.
     Any sharp turns as in $f(x) = |x|$ at $x = 0$, will kill differentiability.
  3. If $f'(a) > 0$, then the function must be increasing there instantaneously.
     If $f'(a) < 0$, then the function must be decreasing there instantaneously.

     If $f'(a) = 0$, then the tangent line is horizontal.
     If $f'(a) = 0$ AND $f''(a) > 0$, then it must be a relative (local) minimum point because the tangent line is horizontal AND the curve is concave up.
     If $f'(a) = 0$ AND $f''(a) < 0$, then it must be a relative (local) maximum point because the tangent line is horizontal AND the curve is concave down.
  3. An inflection point must have $f''(a) = 0$ AND the concavity must change signs.
     Example: $f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$.
     We see that $f''(0) = 0$ and that $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$.

     Example: $f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2$.
     We see that $f''(0) = 0$, but $f''(x) > 0$ on both sides of $x = 0$. So $(0, 0)$ is NOT an inflection point.
  4. If $y = f(x)$ has an inflection point at $(a, f(a))$, then the curve $y = f'(x)$ has either a relative minimum or relative maximum point at $(a, f'(a))$.
     Please review the previous Class Log for an example. That's all the necessary information for curve sketching! If you have any questions, please ask during the next lecture.

- At last, we can learn the "fast" rules for finding derivatives.

  Section 3.1 covers derivatives of polynomials and exponential functions.

  We already know these:
  $$\begin{aligned} [c]' &= 0 \\ [mx + b]' &= m \\ \left[x^2\right]' &= 2x. \end{aligned}$$

  Also, from the properties of limits, we already know that
  $$\begin{aligned} \left[x^2 + 5x\right]' &= \left[x^2\right]' + [5x]' = 2x + 5. \quad \text{(Sum and difference property.)} \\ \left[3x^2\right]' &= 3\left[x^2\right]' = 3(2x) = 6x. \quad \text{(Multiplicative constant property.)} \end{aligned}$$

- We used the algebraic conjugate when we calculated the derivatives with limits:

$$\begin{aligned}\left[\sqrt{x}\right]' &= \left[x^{1/2}\right]' = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}.\\ \left[\frac{1}{\sqrt{x}}\right]' &= \left[x^{-1/2}\right]' = -\frac{1}{2x\sqrt{x}} = -\frac{1}{2x^{3/2}} = -\frac{1}{2}x^{-3/2}.\end{aligned}$$

- We generated a lot of amusing fractions with these:

$$\begin{aligned}\left[\frac{1}{x}\right]' &= \left[x^{-1}\right]' = -\frac{1}{x^2} = (-1)\,x^{-2}.\\ \left[\frac{1}{x^2}\right]' &= \left[x^{-2}\right]' = -\frac{2}{x^3} = -2x^{-3}.\end{aligned}$$

- So it would be nice if we had a general formula for

$$[x^n]' = ???$$

if n is *any real number.*

We do. It's called the General Power Rule (p. 191). It works for any real value of n.

$$[x^n]' = nx^{n-1}.$$

Even though we cannot present the "complete" proof of why this works for all real n, (for example, we have some difficulty deciding how the graph of $f(x) = x^{\sqrt{2}}$ looks...), we can at least show how Newton established this formula for the polynomial functions.

Newton also invented the Binomial Theorem. We showed in class that the coefficients of the following can be determined by Pascal's Triangle:

$(a+b)^1 = \mathbf{1}a + \mathbf{1}b$

$(a+b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2$

$(a+b)^3 = \mathbf{1}a^3 + \mathbf{3}ab^2 + \mathbf{3}a^2b + \mathbf{1}b^3.$

$(a+b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4.$

Here's the general form which works for *any real value* of n.

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ...$$

We note that when n is a positive integer,
this sum stops at $\dfrac{n(n-1)(n-2)\cdots(1)}{n!}a^{n-n}b^n = \mathbf{1}b^n.$

Now let us try to evaluate this derivative:

$$\begin{aligned}[x^n]' &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{x^n + \left(\frac{n}{1!}\right)x^{n-1}h + \left(\text{other terms with at least } h^2 \text{ in them}\right) - x^n}{h}\\ &= \lim_{h\to 0}\frac{nx^{n-1}h + \left(\text{other terms with at least } h^2 \text{ in them}\right)}{h}\end{aligned}$$

So we can factor out an h in the numerator...

$$= \lim_{h\to 0} \frac{h\left(nx^{n-1} + (\text{other terms with at least } h \text{ in them})\right)}{h}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + (\text{other terms with at least } h \text{ in them})\right).$$

All of those "other" terms have at least one factor of h, so by direct substitution, this final limit is equal to nx^{n-1}.

Thus, we have $[x^n]' = nx^{n-1}$, for all positive integers n.

- It actually works for all real values of n. Thus, we have

$$\left[x^{\sqrt{2}}\right]' = \sqrt{2} * x^{\sqrt{2}-1}$$

$$\left[\frac{1}{\sqrt[3]{x}}\right]' = \left[x^{-1/3}\right]' = -\frac{1}{3}x^{-4/3} = -\frac{1}{3x^{4/3}}.$$

- We introduced the Product Rule.

$$(fg)' = fg' + gf'$$

Some students prefer this ordering:

$$(fg)' = fg' + f'g$$

Why? Because there is a neat pattern for larger products...

$$\begin{aligned}(fgh)' &= ((fg)\,h)' \\ &= (fg)\,h' + (fg)'\,h \\ &= fgh' + \left(fg' + f'g\right)h \\ &= fgh' + fg'h + f'gh.\end{aligned}$$

I prefer the first one because I have other formulas which always displays the derivative as the last factor.

- We try NOT to use it if we can conveniently FOIL (distribute) stuff out...

$$\begin{aligned}\left[(x+3)\left(\sqrt{x}-2\right)\right]' &= \left[x\sqrt{x} - 2x + 3\sqrt{x} - 6\right]' \\ &= \left[x^{3/2} - 2x + 3x^{1/2} - 6\right]' \\ &= \frac{3}{2}x^{1/2} - 2 + 3\left(\frac{1}{2}x^{-1/2}\right) + 0 \\ &= \frac{3}{2}x^{1/2} - 2 + \frac{3}{2}x^{-1/2}.\end{aligned}$$

Also, suppose that we would like to solve $f'(x) = 0$. We would probably want to rewrite the expressions with radicals!

$$0 = \frac{3\sqrt{x}}{2} - 2 + \frac{3}{2}\left(\frac{1}{\sqrt{x}}\right)$$

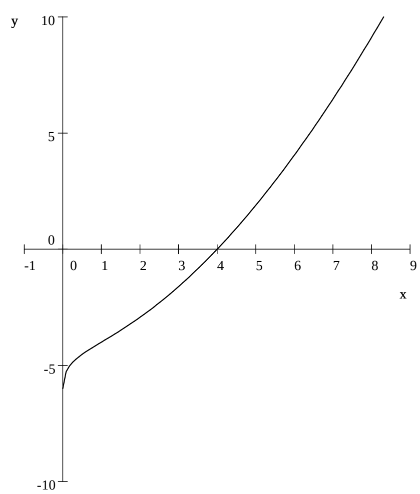
Certainly, we would want to multiply everything by the LCD $= 2\sqrt{x}$.

$$\begin{aligned} 0\left(2\sqrt{x}\right) &= \frac{3\sqrt{x}\left(2\sqrt{x}\right)}{2} - 2\left(2\sqrt{x}\right) + \frac{3\left(2\sqrt{x}\right)}{2}\left(\frac{1}{\sqrt{x}}\right) \\ 0 &= 3x - 4\sqrt{x} + 3. \end{aligned}$$

Let $u = \sqrt{x} \Rightarrow u^2 = x$. This is now a quadratic equation.

$$\begin{aligned} 0 &= 3u^2 - 4u + 3 \\ u &= \frac{4 \pm \sqrt{(-4)^2 - 4(3)(3)}}{2(3)} = \frac{4 \pm \sqrt{-20}}{6}. \end{aligned}$$

No real roots! So we guess that the original function graph has NO horizontal tangent lines anywhere.



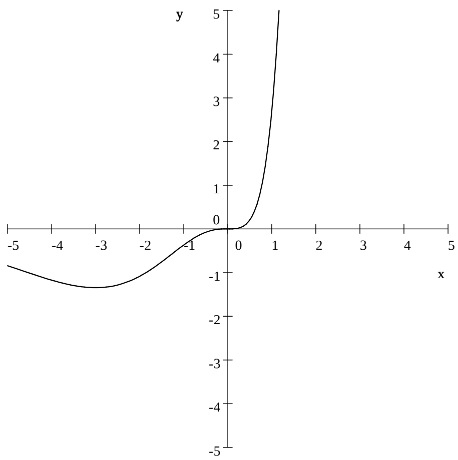
- So we are most likely to use it when polynomials are multiplied by transcendental functions...

$$\left[x^3 e^x\right]' = x^3\left[e^x\right]' + e^x\left[x^3\right]' = x^3 e^x + 3x^2 e^x = \left(x^3 + 3x^2\right) e^x.$$

Can we solve $\left(x^3 + 3x^2\right) e^x = 0$?? Yes!

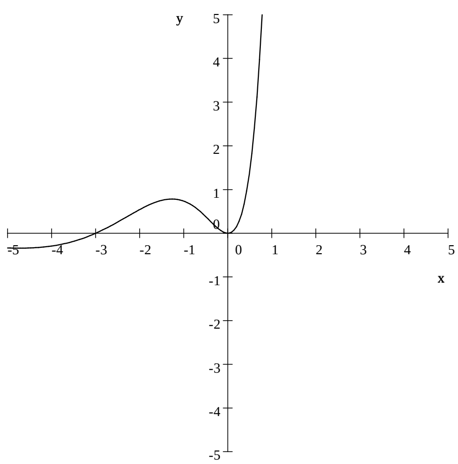
Since e^x is never equal to zero, we can divide both sides by it.

$x^3 + 3x^2 = 0 \Rightarrow x^2(x+3) = 0 \Rightarrow x = -3,\ 0.$



Here is the graph for $y = f(x) = x^3 e^x$.
There is probably an inflection point at $x = 0$.
We see that the tangent line there has slope zero!

There is clearly a relative minimum point at $x = -3$.



Here is the graph for $y' = \left(x^3 + 3x^2\right) e^x$.
The graph of y' crosses the x-axis at $x = -3$,
and is tangent to the x-axis at $x = 0$.