

Class Log for MATH 1401-001 (Calculus I)

- Wednesday, 02/02 [Happy Groundhog Day!]:

We introduced the material from Sections 2.8 and 2.10. For now, we are skipping Section 2.9.

- We reviewed the two definitions of $f'(a)$.

1. This works if we have some algebraic trick available to us which allows us to cancel out the $(x - a)$ in the denominator...

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

2. In general, this one works better...

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

We saw last time, that this definition worked better for $f(x) = \sin(x)$ or $\cos(x)$.

3. If we substitute in x for a (in the last definition), we have a definition for $f'(x)$.

$$[f(x)]' = f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

We verified in class, that $[x^2]' = 2x$ by evaluating this limit.

I handed out some more examples.

$$\text{Try proving that } \left[\frac{1}{\sqrt{x}} \right]' = [x^{-1/2}]' = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}} = -\frac{1}{2x\sqrt{x}}.$$

Later, we will have the Simple Power Rule, which will allow us to evaluate derivatives like that one, very easily!

- We explored other notation systems.

Leibniz [also credited with the invention of differential calculus] notation resembles fractions.

So if $y = f(x)$, then the following are equivalent to $f'(x)$ or y' :

$$\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} [f(x)].$$

The motivation is simple. Leibniz always used the definition of slope whenever he wrote his derivatives.

$$y' = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

The expression $\frac{\Delta y}{\Delta x}$ represents the slope of the secant lines.

We don't use the "D" notation until later in Calc. III. We say D is an operator because we need a function as an input, and we obtain a function as an output. All of the following are operators being applied to $f(x)$:

$$[f(x)]' = D[f(x)] = D_x[f(x)].$$

- We decided that we could repetitively take the derivative. This gives us *second order* derivatives and higher order derivatives.

$$f''(x) = [[f(x)']]' = y'' = \frac{d}{dx} \left[\frac{d}{dx} [f(x)] \right] = \frac{d^2}{dx^2} [f(x)] = \frac{d^2 y}{dx^2}.$$

For example, if $f(x) = mx + b$, then $f'(x) = m$, and $f''(x) = 0$.

If $f(x) = x^2$, then $f'(x) = 2x$, and $f''(x) = 2$.

- So what do we mean when we say that a function is differentiable at $x = a$?

The derivative $f'(a)$ must exist and be a finite real number when $x = a$.

More importantly, when is f NOT differentiable?

1. If f is NOT continuous at $x = a$, then the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ cannot exist.}$$

Example: If f has a removable discontinuity as in

$$f(x) = \frac{x^2 - 4}{x - 2} \text{ at } x = 2.$$

We know that this equivalent to the punctured line $f(x) = x + 2$, $x \neq 2$.

From the left and from the right of $x = 2$, the slope of the line is 1. However, when we attempt to evaluate

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h},$$

we have no value to substitute in for $f(2)$. Thus, $f'(2)$ does not exist.

[If $f(2)$ is defined as some crazy number which again makes f discontinuous at $a = 2$, then we have the situation as described below.]

Another Example: If f has a *jump discontinuity* as in $f(x) = \llbracket x \rrbracket$.

We recall that for $0 \leq x < 1$, the value of $f(x) = 0$, and for $1 \leq x < 2$, the value of $f(x) = 1$.

So f is continuous from the right for $a = 1$, but it is a jump discontinuity since

$$\lim_{x \rightarrow 1^-} f(x) = 0 \neq f(1) = 1.$$

The slope of the line segment as $x \rightarrow 1^-$ is zero. The slope of the next higher line segment as $x \rightarrow 1^+$ is zero, but when we evaluate the left-hand limit (half of the two-sided limit), we obtain

$$\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{0 - 1}{h} = d.n.e.$$

because $(1+h)$ is slightly less than one and the greatest integer function of this value is zero.

[If we have an infinite discontinuity, then it should be obvious that the limit is either *d.n.e* or infinite.]

2. If f takes a “sharp turn” or is not *smooth* at a particular point, then f is not differentiable there.

Consider $y = f(x) = |x|$. We recall the “V” shape. The graph is continuous, but it takes a sharp turn at $x = 0$.

So for $x > 0$, we have $y = x$, and the slope is $(+1)$. For $x < 0$, we have $y = -x$, and the slope is (-1) .

Thus, we have

$$f'(x) = \begin{cases} -1, & x > 0 \\ 1, & x < 0 \\ ??, & x = 0 \end{cases}$$

and we must appeal to the left and right-hand limits. We have

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{|h| - 0}{h} = -1$$

[because h is a small negative number].

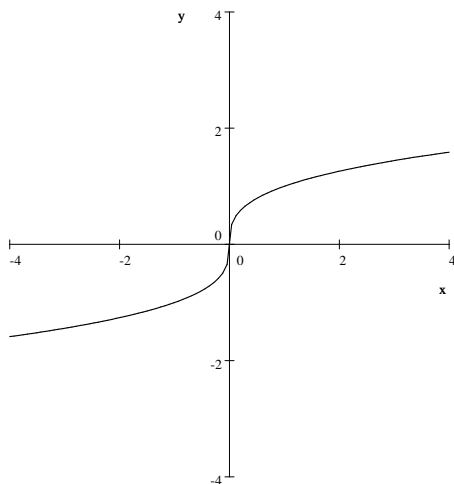
We also have

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = +1$$

[because h is a small positive number]. So the one-sided limits do not agree and the two-sided limit is *d.n.e.*

3. So “smooth” just means that a tangent line exists. If that tangent line is *vertical*, then the derivative is undefined. So if this occurs, then we say that that f is not differentiable where there is a vertical tangent line.

Example: $y = \sqrt[3]{x}$ at $x = 0$. We will examine the derivative formula for this one later.



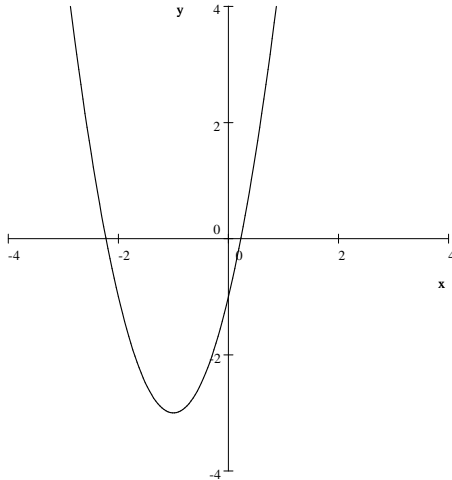
- We always “read” 2-D graphs from left to right.

When f is increasing, then if $f'(a)$ exists, it must be true that $f'(a) > 0$.

Conversely, if f is decreasing, we must have $f'(a) < 0$.

If f has a local minimum (typically, a point at the bottom of a bowl) and the curve is smooth there, then $f'(x)$ at the point should be equal to zero. Furthermore, we have $f'(x) < 0$ just to the left of the local minimum point, and $f'(x) > 0$ just to the right.

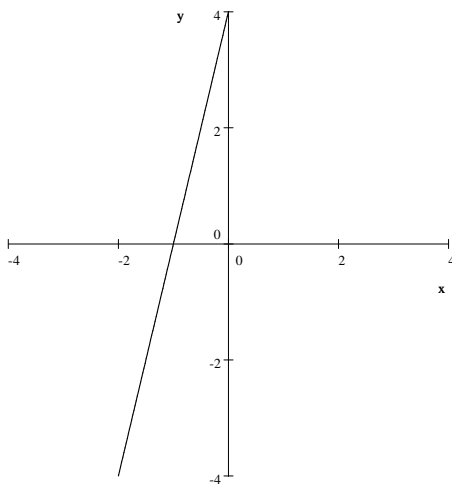
Example: $y = 2x^2 + 4x - 1$



You should try to find this derivative using the limit definitions:

$$[2x^2 + 4x - 1]' = 4x + 4.$$

We see that the vertex of the parabola is located at $x = -1$. This is certainly the relative minimum point: $(-1, -3)$.

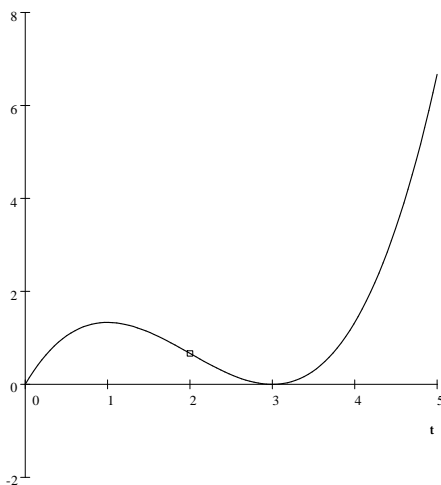


So at the corresponding value of x , we see that $f'(x) = 0$.

For a relative minimum, the graph of f' should go from negative to positive and cross the x-axis at the critical value of x .

The opposite is true of a relative maximum point (typically, the top of a hill). The graph of f' will go from positive to negative and cross the x-axis at the critical value of x .

- If $y = f(t)$ is a position function like $f(t) = \frac{1}{3}t^3 - 2t^2 + 3t$, then its graph is:



We have a relative maximum at $x = 1$ and a relative minimum at $x = 3$.

So the odometer starts at $y = 0$ and then hits a relative maximum point at $(1, \frac{4}{3})$.

We note that our vehicle is instantaneously at rest at that moment, and then reverses direction.

We see that the function is decreasing on the interval $(1, 3)$. The odometer is going “backward” over this interval, and at $t = 3$, the odometer reads $y = 0$ once again.

It turns out that $x = 2$ is an *inflection point*.

What happens at an inflection point?

We have change in concavity! In the previous diagram, the curve is concave down (deceleration) on the interval $[0, 2)$. It is concave up (acceleration) on the interval $(3, +\infty)$. So the first derivative is decreasing until it reaches $t = 2$, and then it begins to increase again.

Try sketching the curve for f' .

It should cross the x-axis at $t = 1$ and $t = 3$, and it should have a relative minimum at $t = 2$.

It should look something like this:

