Class Log for MATH 1401-001 (Calculus I)

- Wednesday, 01/26:

  We finished up the material from Sections 2.3, 2.4., 2.5.

  The definitions in Section 2.4 concerning continuity were most important.

  Functions (and thus, the graphs of these functions) are continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

  In other words, there must be a dot on the curve $y = f(x)$ at $x = a$, AND the curve must converge to that dot from both directions (left and right).

  We then classified the different types of *discontinuities.*

  If the curve (or line) has a puncture in it, then we say the discontinuity is *removable*, because if we redefined $f(a)$ so that the puncture would be filled in, then the curve would be continuous there.

  If the left limit and right limit are both finite but not equal to each other, then it must be a *jump* discontinuity. We defined the "greatest integer function", $f(x) = [\![x]\!]$. This function has jump discontinuities whenever x is an integer.

  If the left limit or right limit or both are not finite (plus or minus infinity), then we have an infinite discontinuity. Typically, there is a vertical asymptote lurking about when this occurs.

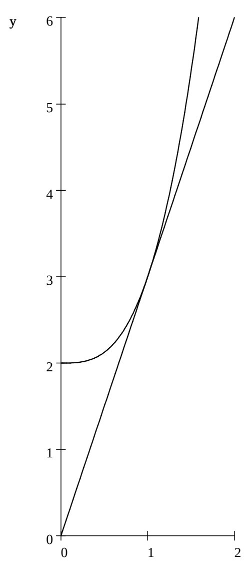
- This leads us to the definitions of *continuity from the left,* and *continuity from the right* (p. 121).

  For example, $f(x) = [\![x]\!]$ is continuous from the right!

  Polynomial and rational functions are always continuous over their domains. When a rational function is undefined, the discontinuity is either infinite or removable.

- We reviewed the Squeeze Theorem and looked at another example: (**#26**) on p. 118.

  If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, then evaluate $\lim_{x\to 1} f(x)$. Here's a sketch:



So the graph of f must fit in those two little wedges which converge upon the point $(1, 3)$.

f must always be above the line $y = 3x$ and below the cubic curve $y = x^3 + 2$.

We verified that $\lim_{x\to 1^-} f(x) = 1$ by the Squeeze Theorem, and that $\lim_{x\to 1^{+-}} f(x) = 1$, also by the Squeeze Theorem.

Thus, the two sided limit is equal to 1.

- The Intermediate Value Theorem only works for continuous functions on an interval $[a, b]$.

  If $f(a) \neq f(b)$, then we can choose any value of $y = k$ between these two values, and we are guaranteed that the horizontal line $y = k$ will intersect the graph of f at least once at $(c, f(c))$, where c is in the interval $[a, b]$.

  For example, since $f(x) = x^3 - 2$ is a polynomial (continuous) function and $f(1) = -1$ and $f(2) = 6$, then we can invoke the Intermediate Value Theorem. We choose the value $y = 0$ which is between (-1) and 6. The theorem guarantees us that there exists $x = c$ such that $f(x) = 0 = x^3 - 2$ on the interval $[1, 2]$.

  We know that the exact answer is $x = \sqrt[3]{2} \doteq 1.2599$ which falls in that interval. Thus, this proves the existence of a real zero of f in the interval $[1, 2]$.

- We briefly talked about how to find the graphs of inverse functions. Effectively, we exchange the x and y-axes.

  We showed that if we restrict the domain of $\tan(x)$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then its range is $(-\infty, +\infty)$. Thus, the domain of $\tan^{-1}(x)$ must be $(-\infty, +\infty)$, and its range must be $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

- We examined "end behaviors" when evaluating

$$\lim_{x \to -\infty} f(x) \quad \textit{or} \quad \lim_{x \to +\infty} f(x).$$

  For example, if $f(x) = x^2$ or $f(x) = \ln(x)$, we know that the right-hand end behavior is $y \to +\infty$. Thus, we use the *extended real numbers*, and include the symbols $-\infty$ and $+\infty$ to represent huge negative and positive numbers.

$$\lim_{x \to +\infty} \ln(x) \infty = +\infty.$$

  If $y = f(x)$ has a horizontal asymptote $y = k$ as $x \to +\infty$, then

$$\lim_{x \to +\infty} f(x) = k.$$

  We showed (from the previous item) that

$$\lim_{x \to +\infty} \tan^{-1}(x) = \frac{\pi}{2} \quad \text{and} \quad \lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

  For rational functions, we apply a trick. Example:

$$\lim_{x \to +\infty} \frac{4x^2 - 356x + 2001}{3x^2 + 60001} = \lim_{x \to +\infty} \frac{\frac{4x^2}{x^2} - \frac{356x}{x^2} + \frac{2001}{x^2}}{\frac{3x^2}{x^2} + \frac{60001}{x^2}}.$$

  The limit of the quotient is the quotient of the limits.

$$\frac{\lim\limits_{x \to +\infty} \left(\frac{4x^2}{x^2} - \frac{356x}{x^2} + \frac{2001}{x^2}\right)}{\lim\limits_{x \to +\infty} \left(\frac{3x^2}{x^2} + \frac{60001}{x^2}\right)} = \frac{\lim\limits_{x \to +\infty} \left(4 - \frac{356}{x} + \frac{2001}{x^2}\right)}{\lim\limits_{x \to +\infty} \left(3 + \frac{60001}{x^2}\right)} = \frac{\lim\limits_{x \to +\infty} (4 - 0 + 0)}{\lim\limits_{x \to +\infty} (3 + 0)} = \frac{4}{3}.$$

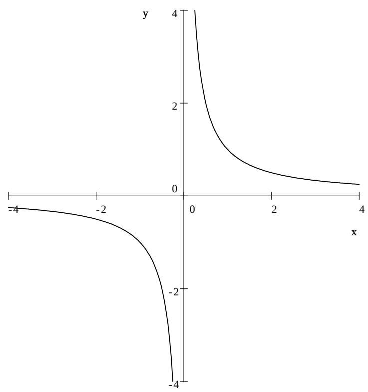
  We choose the highest degree of x represented in the rational fraction.

  Clearly, if the degree of the denominator is equal to the degree of the numerator, then limiting value is the ratio of the leading coefficients (as above). If the degree of the denominator is greater than the degree of the numerator, then the asymptote is $y = 0$. Example:

$$\lim_{x \to +\infty} \frac{3}{x} = 0.$$

- If f has vertical asymptotes, then we must examine the one-sided limits first!

$$\text{We have } \lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty, \text{ but } \lim_{x\to 0} \frac{1}{x} = d.n.e.$$



If the left-hand and right-hand limits are the same infinity, then, for example, we have

$$\lim_{x\to 0^-} \frac{1}{x^2} = +\infty, \quad \lim_{x\to 0^+} \frac{1}{x^2} = +\infty, \text{ and } \lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

See example on pp. 130-131.