

Solutions to Bruce MacMillan's Test #3 – MATH 1401
Spring 2005

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(#1) If $y = (\ln(x))^{\tan(x)}$, then find y' .

Let's do the Logarithmic Differentiation thing again!

$$\ln(y) = \tan(x) * \ln(\ln(x))$$

$$[\ln(y)]' = [\tan(x) * \ln(\ln(x))]'$$

$$\frac{y'}{y} = \tan(x) [\ln(\ln(x))]' + \ln(\ln(x)) [\tan(x)]'$$

$$= \tan(x) \left(\frac{[\ln(x)]'}{\ln(x)} \right) + \ln(\ln(x)) \sec^2(x)$$

$$= \tan(x) \left(\frac{\left(\frac{1}{x} \right)}{\ln(x)} \right) + \ln(\ln(x)) \sec^2(x)$$

$$\frac{y'}{y} = \tan(x) \left(\frac{1}{x \ln(x)} \right) + \ln(\ln(x)) \sec^2(x)$$

Now multiply both sides by y .

$$y' = (\ln(x))^{\tan(x)} \left(\frac{\tan(x)}{x \ln(x)} + \ln(\ln(x)) \sec^2(x) \right).$$

(#2) Suppose $f(x) = 2x + \ln(x)$.

- (a) Find the linear approximation of f at $x = 1$. We have $f(1) = 2(1) + \ln(1) = 2$. We must find the equation of the tangent line at $(1, 2)$.

$$f'(x) = 2 + \frac{1}{x} \Rightarrow f'(1) = 3.$$

The slope of the line is $m = 3$.

$$\begin{aligned} y - 2 &= 3(x - 1) \\ y &= 3x - 1. \end{aligned}$$

It's actually easier to leave it like this:

$$f(x) \approx 3(x - 1) + 2.$$

- (b) Use that equation to approximate $f(1.001)$.

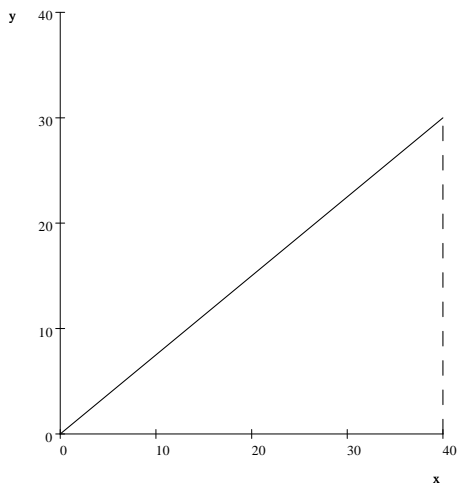
$$\begin{aligned} f(1.001) &\approx 3(1.001 - 1) + 2 \\ &\approx 2.003. \end{aligned}$$

We compare this to $f(1.001) = 2.002 + \ln(1.001) \doteq 2.0029995$. ✓

(#3) Alex has a helium balloon in his hand and Bonnie is standing 40 feet from him.

Alex lets the balloon go, and it rises vertically in the still air. A few seconds later, the balloon is 30 feet in the air and the instantaneous velocity of the balloon is +5 ft/sec upward.

Here's a simple diagram of the triangle.



If y equals the side opposite the standard angle θ , then the hypotenuse is $h = \sqrt{y^2 + 1600}$.

$$\text{Also, } \tan(\theta) = \frac{y}{40} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{40}\right).$$

- (a) How fast is the distance between Bonnie and the balloon increasing when the balloon's height is $y = 30$ feet?

$$h = \sqrt{y^2 + 1600}$$

$$\frac{d}{dt}[h] = \frac{d}{dt}[(y^2 + 1600)^{1/2}]$$

$$\frac{dh}{dt} = \frac{1}{2}(y^2 + 1600)^{-1/2} * \frac{d}{dt}[y^2]$$

$$\frac{dh}{dt} = \frac{1}{2}(y^2 + 1600)^{-1/2} \left(2y * \frac{dy}{dt}\right)$$

$$\frac{dh}{dt} = \frac{y}{\sqrt{y^2 + 1600}} \left(\frac{dy}{dt}\right)$$

The current value of y is 30, and the instantaneous value of $\frac{dy}{dt} = +5$.

$$\frac{dh}{dt} = \frac{30}{\sqrt{30^2 + 1600}} (5) = +3 \text{ ft/sec.}$$

- (b) How fast is the *angle of elevation* (angle between the ground and the line of sight of the balloon) measured from Bonnie's perspective increasing when the balloon's height is $y = 30$ feet? [What are the units of measure in the final answer?]

The units of measure in the final answer must be *radians* per second.

$$\tan(\theta) = \frac{y}{40}$$

$$\frac{d}{dt} [\tan(\theta)] = \frac{1}{40} \frac{d}{dt} [y]$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{40} \left(\frac{dy}{dt} \right).$$

The current value of $\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{50}{40} = \frac{5}{4}$.

$$\left(\frac{5}{4} \right)^2 \frac{d\theta}{dt} = \frac{1}{40} (5) \Rightarrow \frac{d\theta}{dt} = \left(\frac{1}{8} \right) \left(\frac{16}{25} \right) = +\frac{2}{25} \text{ rad/sec.}$$

That's about 4.6° per second.

- (#4) The height and radius of a right circular cylinder are equal, so the volume is $V = \pi h^3$. Suppose that the height (and radius) of the cylinder changes from 10 cm to 10.1 cm.

- (a) Use differentials to approximate the change in the volume of the cylinder.

The current value of $h = 10$ cm. We have $dh = +0.1$ cm.

$$\begin{aligned} d[V] &= \pi * d[h^3] \\ dV &= \pi (3h^2 dh) \\ dV &= \pi (3 * 10^2 * 0.1) = 30\pi \text{ cm}^3. \end{aligned}$$

- (b) Write down the expression for the *relative* change in the volume.

$$\frac{dV}{V} = \frac{30\pi}{(\pi * 10^3)} = \frac{30}{1000} = 3\%.$$

- (#5) Find two positive real numbers whose product is 45, and the sum of the first number plus three times the second number is a minimum.

The two numbers are x and $\frac{45}{x}$.

We wish to minimize $f(x) = x + 3 \left(\frac{45}{x} \right) \Rightarrow f'(x) = 1 + 135 [x^{-1}]' = 1 - \frac{135}{x^2}$.

We set this equal to zero to find the critical number.

$$1 - \frac{135}{x^2} = 0 \Rightarrow x^2 = 135 \Rightarrow x = \sqrt{135} = 3\sqrt{15}.$$

The second number is

$$\frac{45}{3\sqrt{15}} = \frac{15}{\sqrt{15}} = \sqrt{15}.$$

The minimum value of $f(x)$ is

$$\sqrt{15} + 3 \left(\frac{45}{\sqrt{15}} \right) = 10\sqrt{15}.$$

If we perform the Second Derivative Test, we have

$$f''(x) = \left[1 - \frac{135}{x^2} \right]' = \frac{270}{x^3}.$$

This is positive, so our critical value must be a relative (absolute) minimum.

(#6) A rectangular box with an open top and a square base is to be made out of 1200 cm² of material. Find the dimensions of the box that will *maximize* the volume of the box.

Let x = the length of the square base.

If h = the height of the box, then the area of one lateral face is hx .

The surface area of the box (material) must be the base plus the four lateral sides.

$$\text{Surface Area} = 1200 = x^2 + 4hx$$

$$h = \frac{1200 - x^2}{4x}.$$

The volume of the box is $x^2 * h$.

$$f(x) = x^2 \left(\frac{1200 - x^2}{4x} \right) = \frac{1}{4} (1200x - x^3).$$

Find the critical numbers.

$$f'(x) = \frac{1}{4} (1200 - 3x^2) = 0 \Rightarrow x^2 = 400 \Rightarrow x = \pm 20.$$

We want $x = 20$. The largest volume is

$$f(20) = (20^2) \left(\frac{1200 - 20^2}{4(20)} \right) = 4000 \text{ cm}^3.$$

The base is 20 and the height is 10.

We know that this is a maximum from the Second Derivative Test.

$$f''(x) = -\frac{3}{2}x.$$

This is negative when $x = 20$. Thus, we've maximized the volume. [We know that $x = 0$ gives us zero volume.]

(#7) Find the *absolute* minimum and *absolute* maximum values of $f(x) = x^3 - 12x$ on the closed interval $[-3, 5]$.

Find the critical numbers in the interior of the closed interval.

$$f'(x) = 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2.$$

We need the Second Derivative Test.

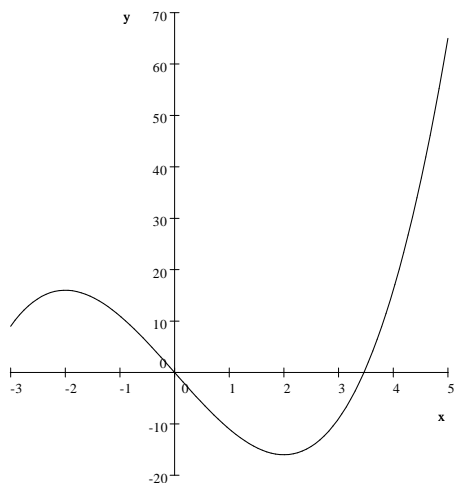
$$f''(x) = 4x.$$

When $x = -2$, we have $f''(-2) = -8$. The point $(-2, f(-2)) = (-2, 16)$ must be a relative maximum point.

When $x = 2$, we have $f''(2) = 8$. The point $(2, f(2)) = (2, -16)$ must be a relative minimum point.

The end points are $(-3, 9)$ and $(5, 65)$.

The absolute minimum point is $(2, -16)$ and the absolute maximum point is $(5, 65)$.



(#8) Let $f(x) = x + \sin(2x)$.

We know that $x = \frac{\pi}{3}$ is a critical number (you can check this!) because $f'(\frac{\pi}{3}) = 0$.

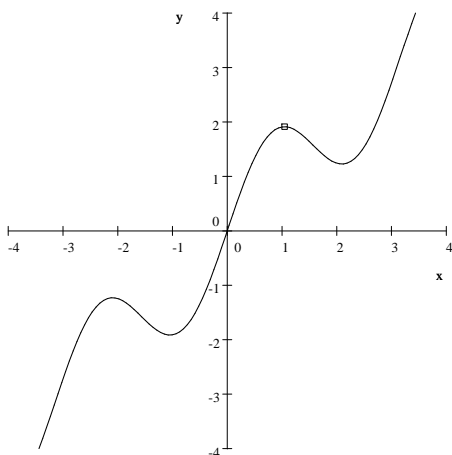
Use the Second Derivative Test to determine if $(\frac{\pi}{3}, \frac{\pi}{3} + \frac{\sqrt{3}}{2})$ is relative minimum point, relative maximum point, or neither.

$$f'(x) = 1 + \cos(2x)[2x]' = 1 + 2\cos(2x)$$

$$f''(x) = 2[\cos(2x)]' = -4\sin(2x)$$

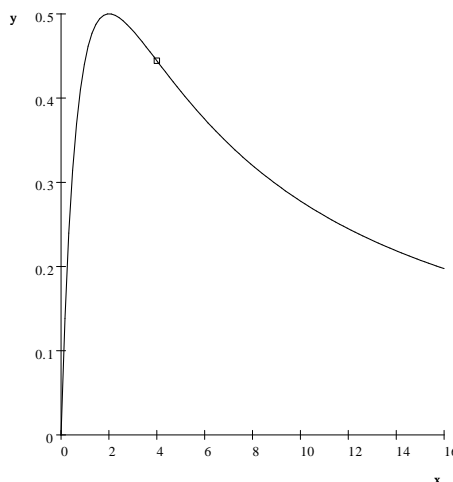
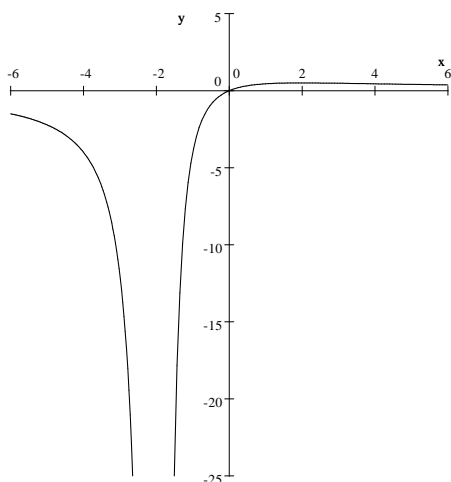
$$f''\left(\frac{\pi}{3}\right) = -4\sin\left(2 * \frac{\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}.$$

It must be relative maximum point.



(#9) Given: $f(x) = \frac{4x}{(x+2)^2}$ $f'(x) = \frac{8-4x}{(x+2)^3}$ $f''(x) = \frac{8x-32}{(x+2)^4}$

Here's a graph. The second one shows the inflection point.



- (a) Write equations for all horizontal and vertical asymptotes of f .

We have a vertical asymptote at $x = -2$.

Since $\lim_{x \rightarrow \infty} \frac{4x}{(x+2)^2} = \lim_{x \rightarrow \infty} \frac{4x}{x^2 + 4x + 4} = 0$ because the degree of the denominator

is larger than the degree of the numerator (divide everything by x^2), we must have a horizontal asymptote at $y = 0$.

- (b) Find all (open) intervals where the f is decreasing.

We see that $f'(x) = 0$ when $x = 2$. This is critical.

There are 3 intervals:

Interval	$8 - 4x$	$(x + 2)^3$	f'	f
$(-\infty, -2)$	+	-	-	Decreasing
$(-2, 2)$	+	+	+	Increasing
$(2, +\infty)$	-	+	-	Decreasing

- (c) Find all (open) intervals where the f is increasing. See table.

(d) Find all points (x, y) where f has local extrema (minima or maxima).

Since $x = 2$ is critical, we need the Second Derivative Test. See below.

When $x = 2$, we have $f''(2) < 0$, so it must be a relative maximum point.

(e) Find all (open) intervals where f is concave down.

We see that $f''(x) = 0$ when $x = 4$. So $x = 4$ might be an inflection point.

Interval	$8x - 32$	$(x + 2)^4$	f''	f
$(-\infty, -2)$	-	+	-	Concave down
$(-2, 4)$	-	+	-	Concave down
$(4, +\infty)$	+	+	+	Concave up

(f) Find all (open) intervals where f is concave up. See table.

(g) Find all points (x, y) where f has inflection points.

Since the concavity change from negative to positive at $x = 4$, we have an inflection

point at $\left(4, \frac{4}{9}\right)$.

(#10) Since $f(x) = \sqrt{2x + 1}$ satisfies the hypothesis of the Mean Value Theorem on the interval $[0, 4]$, find ALL values of x in the interval $(0, 4)$ that satisfy the conclusion of the theorem.

The end points of the interval are $(0, 1)$ and $(4, 3)$.

The average rate of change (AROC) over this interval is $\frac{3 - 1}{4 - 0} = \frac{1}{2}$.

Now find all points in $[0, 4]$ where $f'(x) = \frac{1}{2}$.

$$f'(x) = \left[(2x + 1)^{1/2}\right]' = \frac{1}{2}(2x + 1)^{-1/2} [2x]' = \frac{1}{\sqrt{2x + 1}}.$$

$$\frac{1}{2} = \frac{1}{\sqrt{2x + 1}} \Rightarrow \sqrt{2x + 1} = 2 \Rightarrow 2x + 1 = 4 \Rightarrow x = \frac{3}{2}.$$

(#11) Find each limit if it exists. Show all work. L'Hôpital!

$$(a) \lim_{x \rightarrow 0} \frac{e^{3x} - 3x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{[e^{3x} - 3x - 1]'}{[x^2]'} = \lim_{x \rightarrow 0} \frac{3e^{3x} - 3}{2x}.$$

This is still indeterminate $\left[\frac{0}{0}\right]$.

Do it again.

$$\lim_{x \rightarrow 0} \frac{[3e^{3x} - 3]'}{[2x]'} = \lim_{x \rightarrow 0} \frac{9e^{3x}}{2} = \frac{9}{2}.$$

(b) $\lim_{x \rightarrow +\infty} x * \sin\left(\frac{1}{x}\right)$. This is an indeterminate product $[0 * \infty]$. Rewrite this as a quotient.

$$\lim_{x \rightarrow +\infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow +\infty} \frac{\cos\left(\frac{1}{x}\right) * \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow +\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1.$$

(c) $\lim_{x \rightarrow 0} (e^x + x)^{1/x}$. This is indeterminate $[1^\infty]$.

Suppose that $L = \lim_{x \rightarrow 0} (e^x + x)^{1/x}$ and L is finite.

$$\begin{aligned}\ln(L) &= \lim_{x \rightarrow 0} \ln \left((e^x + x)^{1/x} \right) = \lim_{x \rightarrow 0} \frac{1}{x} \ln(e^x + x) = \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{[\ln(e^x + x)]'}{[x]'} = \lim_{x \rightarrow 0} \frac{\left(\frac{e^x + 1}{e^x + x} \right)}{1} = \lim_{x \rightarrow 0} \frac{e^x + 1}{e^x + x} = \frac{2}{1 + 0} = 2.\end{aligned}$$

Since $\ln(L) = 2$, we have $L = e^2 = e$.

(#12) If $f(x) = 2 \cos(x) - \frac{2}{\sqrt{x}}$, then find the general antiderivative of $f(x)$.

$$\begin{aligned}F(x) &= \int \left(2 \cos(x) - 2x^{-1/2} \right) dx \\ &= 2 \sin(x) - 2 \left(\frac{x^{1/2}}{\left(\frac{1}{2}\right)} \right) + C = 2 \sin(x) - 4\sqrt{x} + C.\end{aligned}$$

(#13) Solve the differential equation: $f'(x) = x + \frac{1}{x} + \frac{1}{x^2}$ and $f(1) = 0$.

Find the general antiderivative.

$$\begin{aligned}F(x) &= \int \left(x + \frac{1}{x} + \frac{1}{x^2} \right) dx \\ &= \int (x + x^{-1} + x^{-2}) dx \\ &= \frac{x^2}{2} + \ln(x) + \frac{x^{-1}}{(-1)} + C \\ F(x) &= \frac{x^2}{2} + \ln(x) - \frac{1}{x} + C.\end{aligned}$$

We need to find C . Substitute in $x = 1$.

$$\begin{aligned}F(1) &= 0 = \frac{1^2}{2} + \ln(1) - \frac{1}{1} + C \\ 0 &= \frac{1}{2} + 0 - 1 + C \\ C &= \frac{1}{2}.\end{aligned}$$

The final solution is $F(x) = \frac{x^2}{2} + \ln(x) - \frac{1}{x} + \frac{1}{2}$.