

DEBRUIJN-LIKE SEQUENCES AND THE IRREGULAR CHROMATIC NUMBER OF PATHS AND CYCLES

MICHAEL FERRARA, CHRISTINE LEE, PHIL WALLIS
DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES
UNIVERSITY OF COLORADO DENVER
ELLEN GETHNER
DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF COLORADO DENVER

ABSTRACT. A *deBruijn sequence of order k* , or a k -deBruijn sequence, over an alphabet \mathcal{A} is a sequence of length $|\mathcal{A}|^k$ in which the last element is considered adjacent to the first and every possible k -tuple from \mathcal{A} appears exactly once as a string of k -consecutive elements in the sequence. We will say that a cyclic sequence is *deBruijn-like* if for some k , each of the consecutive k -element substrings is distinct.

A vertex coloring $\chi : V(G) \rightarrow [k]$ of a graph G is said to be *proper* if no pair of adjacent vertices in G receive the same color. Let $C(v; \chi)$ denote the multiset of colors assigned by a coloring χ to the neighbors of vertex v . A proper coloring χ of G is *irregular* if $\chi(u) = \chi(v)$ implies that $C(u; \chi) \neq C(v; \chi)$. The minimum number of colors needed to irregularly color G is called the *irregular chromatic number* of G . The notion of the irregular chromatic number pairs nicely with other parameters aimed at distinguishing the vertices of a graph. In this paper, we demonstrate a connection between the irregular chromatic number of cycles and the existence of certain deBruijn-like sequences. We then determine exactly the irregular chromatic number of C_n and P_n for $n \geq 3$, thus verifying two conjectures given by Okamoto, Radcliffe and Zhang.

Keywords: deBruijn Sequence, Irregular Chromatic Number, Distinguishing

1. INTRODUCTION

1.1. Terminology and Notation. In this paper we consider only simple graphs, namely those graphs with no loops and no multiple edges. We allow P_n and C_n to denote the path and cycle of order n , respectively. Additionally, $[k]$ will denote the set $\{1, \dots, k\}$ and if D is a digraph $A(D)$ will signify the arc set of D . Let $n_d(G)$ denote the number of vertices of a graph G with degree d . For any undefined terms, we refer the reader to [9].

A k -coloring of a graph G is an assignment of labels (“colors”) from the set $[k]$ to the vertices of G . A vertex coloring $\chi : V(G) \rightarrow [k]$ of a graph G is said to be *proper* if no pair of adjacent vertices in G receive the same color. The *chromatic number* of a graph G , denoted $\chi(G)$, is the smallest k such that there exists a proper k -coloring of G .

1.2. DeBruijn-Like Sequences. A *deBruijn sequence of order k* , or a k -deBruijn sequence, over an alphabet \mathcal{A} is a sequence of length $|\mathcal{A}|^k$ in which the last element is considered adjacent to the first and every possible k -tuple from \mathcal{A} appears exactly once as a string of k -consecutive elements in the sequence. We will say that a cyclic sequence is *deBruijn-like* if for some k , each of the consecutive k -element substrings is distinct.

In part due to the connection between deBruijn sequences and DNA sequencing, the problem of finding deBruijn-like sequences with additional structure has received an increased amount of attention recently. A good example of this is [4] in which the authors describe a graphical approach to generating deBruijn-like sequences where each consecutive k -element substring is distinct up to reversal and complementation (in the same way that the DNA base pairs C-G and A-T are complementary). In this paper, we construct deBruijn-like sequences that allow us to approach a specific problem in graph coloring, described in the next section. DeBruijn-like sequences (of any type) are also of interest as purely combinatorial objects, independent of their various applications.

1.3. Irregular Vertex Colorings. The problem of developing a method by which one can distinguish the vertices of a graph has been of interest for some time. Most examples of such methods generally involve labeling the edges or vertices of a graph in a manner that allows for differentiation of arbitrary pairs of vertices. Some examples include (but are certainly not limited to) the irregularity strength of a graph, first introduced in [10], distance codes [13, 18] and vertex distinguishing edge colorings, introduced in [7].

The following method for differentiating vertices is of interest here. In [2], the authors defined a *distinguishing coloring* of a graph G to be a (not necessarily proper) coloring of the vertices of G that is preserved only by the trivial automorphism. The authors furthermore defined the *distinguishing number* of G to be the minimum number of colors in a distinguishing coloring. This parameter has been widely studied (for two nice examples, see [1] and [14]) and was quite recently extended to consider distinguishing colorings of a graph that are also proper vertex colorings. In [11], the authors define the *distinguishing chromatic number* of a graph G , denoted $\chi_D(G)$, to be the minimum number of colors in a *proper* distinguishing coloring.

In this paper we consider a variant of $\chi_D(G)$, first introduced in [17]. Let $C(v; \chi)$ denote the multiset of colors assigned by a coloring χ to the neighbors of vertex v . A proper coloring χ of G is *irregular* if $\chi(u) = \chi(v)$ implies that $C(u; \chi) \neq C(v; \chi)$. The minimum number of colors needed to irregularly color G is called the *irregular chromatic number* of G . Clearly, any irregular coloring of a graph G is also a distinguishing coloring, and it is therefore clear that $\chi(G) \leq \chi_D(G) \leq \chi_{irr}(G)$. However, $\chi_{irr}(G)$ may not provide a very good upper bound on $\chi_D(G)$. In fact, these parameters may be arbitrarily far apart.

Given an irregular k -coloring χ of a graph G and a vertex v in G , $C(v)$ will be a multiset of cardinality $d(v)$ with elements chosen from the $k - 1$ colors distinct from $\chi(v)$. We recall here that the number of d -element multisets over an alphabet of $k - 1$ symbols is $\binom{k+d-2}{d}$ and note that this implies there can be at most $\binom{k+d-2}{d}$

vertices of degree d in G that are assigned the same color by χ . This observation was made in the following lemma from [17].

Lemma 1. *Let G be a graph and let k be a positive integer. If*

$$n_d(G) > k \binom{k+d-2}{d}$$

then

$$\chi_{irr}(G) \geq k + 1.$$

Lemma 1 serves to underscore the distinction between determining $\chi(G)$ and $\chi_{irr}(G)$. For instance, the widely heralded Four Color Theorem [5, 16] states that any planar graph can be properly colored with at most four colors. However, as the average degree of any planar graph is strictly less than six, Lemma 1 implies that if $\{G_n\}_{n \geq 1}$ is an infinite sequence of planar graphs where each G_i has order i , then $\lim_{n \rightarrow \infty} \chi_{irr}(G_n) = \infty$.

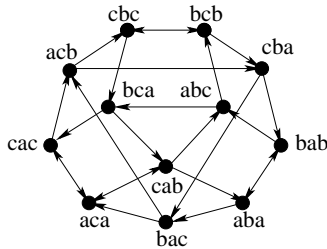
As another example, $\chi(P_n) = \chi(C_{2n}) = 2$ and $\chi(C_{2n-1}) = 3$ for all $n \geq 2$. However, Lemma 1 implies that the irregular chromatic number of these graphs is unbounded as n tends to infinity. These graphs also allow us to contrast χ_{irr} and χ_D , as in [11] it was shown that for all n , $\chi_D(P_n) \leq 3$ and that $\chi_D(C_n) \leq 4$. This is undoubtedly due to the fact that in computing $\chi_{irr}(G)$ we wish to distinguish the vertices of G locally, while χ_D affords the flexibility of a more global perspective. This example also demonstrates that the added condition of irregularity may complicate the coloring of some graphs for which determining χ or χ_D is quite easy.

In this paper, we determine exactly the irregular chromatic number of an arbitrary path or cycle. A similar question was considered [6] in the context of vertex-distinguishing edge colorings, and the similarities and differences between the approaches there and those utilized in this paper help to underscore the differences between these parameters. In the next section, we define a combinatorial structure that allows us to relate irregular colorings of paths and cycles to the existence of certain types of deBruijn-like sequences.

2. THE OVERLAP DIGRAPH $\mathcal{D}_3(\mathcal{A})$

Given an alphabet \mathcal{A} , define $\mathcal{D}_3(\mathcal{A})$ to be the digraph that has as its vertex set the collection of ordered three element strings $a_1a_2a_3$ from \mathcal{A} with the property that $a_1 \neq a_2$ and $a_2 \neq a_3$. In general we will refer to these three element strings, and hence the vertices of $\mathcal{D}_3(\mathcal{A})$, as *words* over the alphabet \mathcal{A} . If $u = u_1u_2u_3$ and $v = v_1v_2v_3$ are vertices in $V(\mathcal{D}_3(\mathcal{A}))$, then uv is in $A(\mathcal{D}_3(\mathcal{A}))$ if and only if $u_2 = v_1$ and $u_3 = v_2$. Since $\mathcal{D}_3(\mathcal{A}_1)$ and $\mathcal{D}_3(\mathcal{A}_2)$ are isomorphic if and only if $|\mathcal{A}_1| = |\mathcal{A}_2|$, we will let $\mathcal{D}_3(k)$ denote the overlap digraph constructed from an arbitrary k -element alphabet. If $\mathcal{A} = \{a_1, \dots, a_n\}$ we may denote $\mathcal{D}_3(\mathcal{A})$ by $\mathcal{D}_3(a_1, \dots, a_n)$. The digraph $\mathcal{D}_3(\mathcal{A})$ is an example of an *overlap digraph*, and is in fact an induced subgraph of the classical 3-deBruijn digraph [12].

Our proof techniques will repeatedly utilize the structure of $\mathcal{D}_3(\mathcal{A})$ when $|\mathcal{A}| = 3$, as depicted in Figure 1.

FIGURE 1. $\mathcal{D}_3(a, b, c)$

The link between overlap digraphs of this type and irregular coloring of paths and cycles is the focus of the results in this paper.

2.1. Overlap Digraphs and Irregular Colorings. For any word $w = a_i a_j a_k$, define the *conjugate* of w to be the word $\bar{w} = a_k a_j a_i$. If $\bar{w} = w$ we then say that w is *self-conjugate* or is a *palindrome*. A (directed) cycle C in $\mathcal{D}_3(\mathcal{A})$ is *conjugate-avoiding* if for all $w \in V(C)$, $\bar{w} \in V(C)$ if and only if w is self-conjugate. Similarly, a (directed) path P is conjugate-avoiding if for every vertex w of degree 2 in P , \bar{w} is also a vertex of degree 2 in P if and only if w is self-conjugate. In much the same way that each k -deBruijn sequence corresponds to an eulerian circuit in the classical k -deBruijn digraph over \mathcal{A} [12], each conjugate avoiding cycle can be associated with a deBruijn-like sequence. The consecutive 3-element substrings of this sequence correspond to a collection of words, distinct up to conjugation, in which no two letters appear consecutively.

Let $V(C_n) = \{v_1, \dots, v_n\}$ where the vertices are listed in the order they appear on the cycle, and let $\chi : V(C_n) \rightarrow \mathcal{A}$ be an irregular coloring of C_n . For each vertex v_i the string $w_i = \chi(v_{i-1})\chi(v_i)\chi(v_{i+1})$, where the indices are taken modulo n , is a vertex in $\mathcal{D}_3(\mathcal{A})$. Since χ is an irregular coloring of C_n , for any two distinct words w_i and w_j we have that $w_i \neq w_j$ and that $w_i \neq \bar{w}_j$. Indeed, if this were not the case, the vertices v_i and v_j would be assigned the same color by χ and their neighbors would also be assigned the same multiset of colors by χ . This would imply that v_i and v_j have the same color code, contradicting the irregularity of χ . In fact, it is not difficult to see that for each i , $w_i w_{i+1}$ is an arc in $\mathcal{D}_3(\mathcal{A})$ and, moreover, the vertices w_1, \dots, w_n in this order form a cycle C .

By the previous discussion, the irregularity of χ implies that C is conjugate-avoiding. An identical analysis yields that if C is a conjugate-avoiding cycle of length n in $\mathcal{D}_3(k)$, then this cycle gives an irregular coloring of C_n using at most k colors. The following lemma summarizes these observations and an example is given in Figure 2.

Lemma 2. *Let n and $k \geq 3$ be positive integers. Then $\chi_{\text{irr}}(C_n) \leq k$ if and only if $\mathcal{D}_3(k)$ contains a conjugate-avoiding cycle of length n .*

The problem of determining the irregular chromatic number of P_n is similar to that of determining the irregular chromatic number of C_n , but is somewhat complicated by the presence of the two end-vertices. Again, let $V(P_n) = \{v_1, \dots, v_n\}$ where the vertices are indexed by the order they appear on the path and let

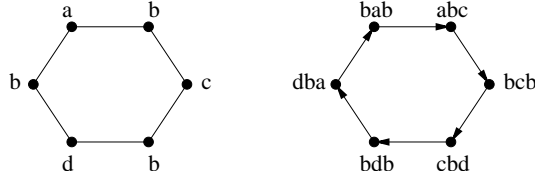


FIGURE 2. An irregular coloring of C_6 and the associated conjugate-avoiding cycle in $\mathcal{D}_3(4)$

$\chi : V(P_n) \rightarrow \mathcal{A}$ be an irregular coloring of P_n . For each vertex v_i where $2 \leq i \leq n-1$ (the vertices of degree 2 in P_n), the string $w_i = \chi(v_{i-1})\chi(v_i)\chi(v_{i+1})$ represents a vertex in $\mathcal{D}_3(\mathcal{A})$. Since χ is an irregular coloring, each of these w_i must be distinct and for each w_i and w_j we must have that $w_i \neq \overline{w_j}$. For each i with $1 \leq i \leq n-1$ we have that $w_i w_{i+1}$ is an arc in $\mathcal{D}_3(\mathcal{A})$ which implies that the vertices w_i , in order, form a conjugate-avoiding path of length $n-2$ in $\mathcal{D}_3(\mathcal{A})$.

Prior to discussing those conjugate-avoiding paths in $\mathcal{D}_3(\mathcal{A})$ that can be used to generate irregular colorings of paths, we need to consider the color codes of v_1 and v_n , which are the vertices of degree one in P_n . Since χ is an irregular coloring, either $\chi(v_1) \neq \chi(v_n)$ or $\chi(v_2) \neq \chi(v_{n-1})$ (or possibly both). We define a conjugate-avoiding path from $w_1 = a_1 a_2 a_3$ to $w_k = a_k a_{k+1} a_{k+2}$ in $\mathcal{D}_2(\mathcal{A})$ to be *boundary distinguished* if either $a_1 \neq a_{k+2}$ or $a_2 \neq a_{k+1}$. This additional condition allows us to characterize those conjugate-avoiding paths in $\mathcal{D}_3(\mathcal{A})$ that give rise to irregular colorings of P_n .

Lemma 3. *Let $k \geq 3$ and n be positive integers. Then $\chi_{irr}(P_n) \leq k$ if and only if $\mathcal{D}_3(k)$ contains a boundary distinguished conjugate-avoiding path on $n-2$ vertices.*

In the following sections, we will construct conjugate-avoiding paths and cycles in $\mathcal{D}_3(k)$ and use them to determine the irregular chromatic number of paths and cycles.

3. IRREGULAR COLORINGS OF CYCLES

The irregular chromatic number of paths and cycles was discussed in both [15] and [17]. For the remainder of this paper, we let n_k denote the quantity $k \binom{k}{2}$. The main result of this section is as follows.

Theorem 1. *Let $k \geq 4$ and $n_{k-1} + 1 \leq n \leq n_k$ be integers. Then*

$$\chi_{irr}(C_n) = \begin{cases} k & \text{if } n \neq n_k - 1 \\ k + 1 & \text{if } n = n_k - 1. \end{cases}$$

We note that Theorem 1 verifies a conjecture given in [15].

Proof. Let k and n , where $n_{k-1} + 1 \leq n \leq n_k$, be as given in the statement of the theorem. In light of Lemma 2 we note that via an elementary counting argument, one can see that the maximum size of a conjugate-free subset of $V(\mathcal{D}_3(k-1))$, and hence the maximum possible length of a conjugate-avoiding cycle in $\mathcal{D}_3(k-1)$,

is n_{k-1} . Consequently, as $n > n_{k-1}$ it must be that $\chi_{irr}(C_n) \geq k$. In line with Lemma 2, the proof proceeds by constructing conjugate-avoiding cycles of lengths $3, \dots, n_k - 2$ and n_k in $\mathcal{D}_3(k)$.

Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be a k -element alphabet. Each of the 3-element subsets of \mathcal{A} can be associated with a unique subgraph of $\mathcal{D}_3(\mathcal{A})$ that is isomorphic to $\mathcal{D}_3(3)$. The subgraphs associated with the 3-element subsets S_i and S_j intersect if and only if $|S_i \cap S_j| = 2$.

With this observation in mind, we define the *auxiliary graph* of $\mathcal{D}_3(k)$, denoted $G_{aux}(k)$, to be the graph whose vertices are 3-element subsets of \mathcal{A} such that two vertices are adjacent if and only if their corresponding sets share exactly two elements. The structure of $G_{aux}(k)$ mimics the structure of $\mathcal{D}_3(k)$, where the vertices of $G_{aux}(k)$ represent distinct copies of $\mathcal{D}_3(3)$ in $\mathcal{D}_3(k)$.

For a vertex v in $V(G_{aux}(k))$, we define the *preimage* of v to be the copy of $\mathcal{D}_3(3)$ in $\mathcal{D}_3(k)$ that is associated with v . Similarly, for any subgraph H of $G_{aux}(k)$, we define the preimage of H to be the subgraph of $\mathcal{D}_3(k)$ spanned by the preimages of the elements of $V(H)$.

It will be useful to more closely examine the structure of $\mathcal{D}_3(3)$, with Figure 1 serving as a helpful reference. The vertices of $D = \mathcal{D}_3(a, b, c)$ can be partitioned into the two directed triangles abc, bca, cab and cba, bac, acb , henceforth referred to as the *central triangles* of D , along with three pairs of palindromes, each determined by a 2-element subset of $\{a, b, c\}$.

The next claim allows us to use the structure of $G_{aux}(k)$ to construct cycles in $\mathcal{D}_3(k)$ using only the vertices in central triangles in some or all of the copies of $\mathcal{D}_3(3)$. For convenience, we extend the definition of a central triangle as follows. If a conjugate-avoiding cycle C in $\mathcal{D}_3(k)$ has the property that $V(C) = V(\tau_1) \cup V(\tau_2) \cup \dots \cup V(\tau_t)$, where each τ_i is a central triangle in some copy of $\mathcal{D}_3(3)$, then we will say that C is *central*. Clearly, if C is central, then $|V(C)| = 3t$ for some integer t . Also, each of the triangles τ_i must be selected from distinct copies of $\mathcal{D}_3(3)$ in $\mathcal{D}_3(k)$, as C is conjugate-avoiding. For clarity, we point out that while $V(C)$ is the disjoint union of the sets $V(\tau_i)$, if $|V(C)| \geq 3$ it will not be the case that $E(C)$ is the disjoint union of the sets $E(\tau_i)$.

Claim 1. *Let $k \geq 3$ and t be integers such that $1 \leq t \leq \binom{k}{3}$. Then there is a conjugate-avoiding cycle C of length $3t$ in $\mathcal{D}_3(k)$ such that C is central.*

Proof. Let T be any spanning tree of $G_{aux}(k)$, and define a sequence of trees $T_1 \subset \dots \subset T_{\binom{k}{3}}$ such that each T_i has order i and $T_{\binom{k}{3}} = T$. We will show that there is a central conjugate-avoiding cycle of length $3t$ in the preimage of T_t for each $1 \leq t \leq \binom{k}{3}$.

We proceed by induction on t , and note that the claim holds when $t = 1$. Thus, let C be a central conjugate-avoiding cycle in the preimage of T_t for $t \geq 1$ and let $v = V(T_{t+1}) \setminus V(T_t)$. Furthermore, let u be a neighbor of v in T_{t+1} . Assume that the preimage of v is $\mathcal{D}_3(a_1, a_2, a_3)$ and that the preimage of u is, without loss of generality, $\mathcal{D}_3(a_1, a_2, x)$. Since C is central, the vertices of one of the central triangles of $\mathcal{D}_3(a_1, a_2, x)$ are also vertices in $V(C)$. Without loss of

generality, assume that the vertices a_1a_2x, a_2xa_1 and xa_1a_2 are in $V(C)$. Then there is some y (possibly equal to x) such that the arc from xa_1a_2 to a_1a_2y is in C . We will augment C by removing the arc from xa_1a_2 to a_1a_2y and adding the arcs $(xa_1a_2, a_1a_2a_3), (a_1a_2a_3, a_2a_3a_1), (a_2a_3a_1, a_3a_1a_2)$ and $(a_3a_1a_2, a_1a_2y)$.

As C was central and conjugate-avoiding, this results in a central conjugate-avoiding cycle of length $3t + 3$ in the preimage of T_{t+1} . \square

To demonstrate the augmentation process described in Claim 1, Figure 3 shows a central triangle in $\mathcal{D}_3(a, b, c)$ extended to a central 6-cycle in the subgraph of $\mathcal{D}_3(4)$ induced by $\mathcal{D}_3(a, b, c)$ and $\mathcal{D}_3(a, b, d)$.

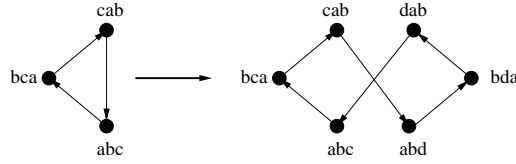


FIGURE 3. Constructing a central cycle of length six from a central cycle of length three

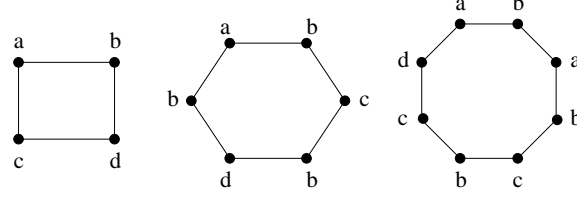
We now show that it is possible to increase the lengths of central cycles, and many other types of cycles, through the addition of pairs of self-conjugate vertices.

Claim 2. *Let A be an alphabet and suppose a_1, a_2 and x are distinct elements in A . If C is a conjugate-avoiding cycle in $\mathcal{D}_3(A)$ that contains the vertex a_1a_2x but does not contain the palindromes $a_1a_2a_1$ and $a_2a_1a_2$, then there is a conjugate-avoiding cycle C' in $\mathcal{D}_3(A)$ such that $V(C') = V(C) \cup \{a_1a_2a_1, a_2a_1a_2\}$.*

Proof. Let C be as given, and let $y \in A$ be such that ya_1a_2 is the predecessor of a_1a_2x in C . We construct C' by removing the arc from ya_1a_2 to a_1a_2x from C and adding the arcs $(ya_1a_2, a_1a_2a_1), (a_1a_2a_1, a_2a_1a_2)$ and $(a_2a_1a_2, a_1a_2x)$. \square

Claim 2 allows us to construct conjugate-avoiding cycles of length $3t, 3t + 2$ and $3t + 4$ for $1 \leq t \leq \binom{k}{3}$ in $\mathcal{D}_3(k)$ by adding either one or two pairs of palindromes to the conjugate-avoiding cycle of length $3t$ assured by Claim 1. It is not difficult to see that this implies that $\mathcal{D}_3(k)$ contains conjugate-avoiding cycles of length ℓ for any ℓ between 3 and $3\binom{k}{3}$ except for $\ell = 4, 6$ or 8. Inspection of Figure 1 yields that $\mathcal{D}_3(3)$ does not contain conjugate-avoiding cycles of length 4, 6 or 8, while the irregular colorings given in Figure 4 demonstrate that cycles of this length do appear in $\mathcal{D}_3(k)$ for $k \geq 4$.

We now wish to demonstrate the existence of cycles of length $3\binom{k}{3}, \dots, n_k - 2$ and n_k in $\mathcal{D}_3(k)$. Let C and C' denote central conjugate-avoiding cycles in $\mathcal{D}_3(k)$ of length $3\binom{k}{3}$ and $3\binom{k}{3} - 3$ respectively. Since C is a central cycle of maximum length, we note that C contains the vertices of a central triangle from each copy of $\mathcal{D}_3(3)$ contained in $\mathcal{D}_3(k)$. Furthermore, note that C' contains the vertices of a central triangle from all but exactly one of the copies of $\mathcal{D}_3(3)$ contained in $\mathcal{D}_3(k)$. This implies that both C and, since $k \geq 4$, C' contain vertices of the form a_1a_2x for every choice of a_1 and a_2 in an appropriate k -element alphabet A .

FIGURE 4. Irregular 4-colorings of C_4 , C_6 and C_8

We may therefore proceed to augment both C and C' using Claim 2. By adding each of the $\binom{k}{2}$ pairs of palindromes in $\mathcal{D}_3(k)$ to C , it is possible to create conjugate-avoiding cycles of length $3\binom{k}{3} + 2, 3\binom{k}{3} + 4, \dots, 3\binom{k}{3} + 2\binom{k}{2} = n_k$ in $\mathcal{D}_3(k)$. Similarly, by adding these pairs of palindromes to C' , it is possible to create conjugate-avoiding cycles of length $3\binom{k}{3} - 1, 3\binom{k}{3} + 1, \dots, 3\binom{k}{3} + 2\binom{k}{2} - 3 = n_k - 3$. It is not difficult to see that this demonstrates the existence of conjugate-avoiding cycles of lengths $3\binom{k}{3}, \dots, n_k - 2$ and n_k in $\mathcal{D}_3(k)$. Together with the shorter cycles constructed above, this demonstrates that $\mathcal{D}_3(k)$ contains conjugate-avoiding cycles of length $3, \dots, n_k - 2$ and n_k .

We complete the proof by noting that Theorem 3.3 from [17] shows that for any $k \geq 3$, $\chi_{irr}(C_{n_k-1}) \geq k+1$. This implies that $\mathcal{D}_3(k)$ contains no conjugate-avoiding cycle of length $n_k - 1$. We have shown, however, that there is a conjugate-avoiding cycle of length $n_k - 1$ in $\mathcal{D}_3(k+1)$ and hence that $\chi_{irr}(C_{n_k-1}) = k+1$. \square

For completeness, we state the following.

Corollary 2. *Let $n \geq 3$ and $k \geq 3$ be integers, and let $n_k = k\binom{k}{2}$. Then*

$$\chi_{irr}(C_n) = \begin{cases} 3 & n = 3, 5, 7, 9 \\ 4 & n = 4, 6, 8 \\ k & k \geq 4 \text{ and either } n_{k-1} + 1 \leq n \leq n_k - 2 \text{ or } n = n_k \\ k+1 & k \geq 4 \text{ and } n = n_k - 1. \end{cases}$$

4. IRREGULAR COLORINGS OF PATHS

In this section, we are interested in determining $\chi_{irr}(P_n)$ for all $n \geq 2$. It is not difficult to see that $\chi_{irr}(P_2) = \chi_{irr}(P_4) = 2$ and that $\chi_{irr}(P_3) = 3$. For $n \geq 5$ we will utilize Lemma 3 and demonstrate the existence of boundary distinguished conjugate-avoiding paths of order $n - 2$ in $\mathcal{D}_3(k)$ for an appropriate choice of k .

The main result of this section is as follows, and serves to verify a conjecture given in [15].

Theorem 3. *Let $k \geq 3$ and n be integers such that $n_{k-1} + 3 \leq n \leq n_k + 2$ be integers. Then*

$$\chi_{irr}(P_n) = k.$$

Proof. For $5 \leq n \leq 11$, Theorem 3 can be easily verified by inspection of Figure 1, which yields boundary distinguished conjugate-avoiding paths of length $3, \dots, 9$ in $\mathcal{D}_3(3)$. For $n \geq 12$, and hence $k \geq 4$, Theorem 3 follows almost immediately

from Theorem 1 and the following lemma, which appears as Proposition 4.3 in [15]. We give an alternate proof here that utilizes the observations made above relating $\chi_{irr}(P_n)$ to the structure of $\mathcal{D}_3(k)$.

Lemma 4. *For each integer $n \geq 5$, $\chi_{irr}(P_n) \leq \chi_{irr}(C_{n-2})$.*

Proof. Let n be as given, and let $\chi_{irr}(C_{n-2}) = t$. Let C be the conjugate-avoiding cycle in $\mathcal{D}_3(t)$ associated with some irregular t -coloring of C_{n-2} and let $a_1a_2a_3$ and a_2a_3x be consecutive vertices on C . Let P be the path in $\mathcal{D}_3(t)$ obtained by deleting the arc $(a_1a_2a_3, a_2a_3x)$ from C . As C is conjugate-avoiding, so too is P , and since $a_2 \neq a_3$, we conclude that P is boundary distinguished. Thus P is a boundary distinguished conjugate-avoiding path of order $n - 2$ in $\mathcal{D}_3(t)$, implying that $\chi_{irr}(P_n) \leq t$. \square

It was shown in Theorem 1 that when $k \geq 4$, $\mathcal{D}_3(k)$ contains conjugate-avoiding cycles of length $3, \dots, n_k - 2$ and n_k . This, along with Lemma 4, implies that if $5 \leq n \leq n_k$ or if $n = n_k + 2$ then $\chi_{irr}(P_n) \leq k$.

It remains to show that if $n = n_k + 1$ then $\chi_{irr}(P_n) \leq k$. Indeed, let C be a conjugate-avoiding cycle of length n_k in $\mathcal{D}_3(k)$ and let $a_1a_2a_3, a_2a_3a_4$ and $a_3a_4a_5$ be three consecutive vertices on C . Removing $a_2a_3a_4$ from C will result in a conjugate-avoiding path P and P will be boundary distinguished provided $a_2 \neq a_4$. That is, the removal of any non-palindrome from C will result in a boundary distinguished conjugate-avoiding path of order $n_k - 1$ in $\mathcal{D}_3(k)$, implying that $\chi_{irr}(P_{n_k+1}) \leq k$.

In particular, the preceding analysis yields that if $n_{k-1} + 3 \leq n \leq n_k + 2$ then $\chi_{irr}(P_n) \leq k$. Lemma 1 (with $r = 2$) implies that for these values of n , $\chi_{irr}(P_n) \geq k$. The result follows. \square

5. CONCLUSION

It would be of interest to extend the class of conjugate-free deBruijn-like sequences to include those cyclic sequences whose t -element substrings are distinct up to conjugation for some $t > 3$. At this time, we do not have a particular application of these sequences in mind, but it is reasonable to think that one may be found.

The notion of an irregular vertex coloring of a graph is also relatively new, and we believe that the area remains ripe for fruitful investigations in the future. A natural, and perhaps approachable, extension of the results in this paper would be to determine the irregular chromatic number of an arbitrary graph G with $\Delta(G) \leq 2$; that is, G is an arbitrary union of paths and cycles. Let G be the graph whose components are the paths P_{i_1}, \dots, P_{i_k} and the cycles C_{j_1}, \dots, C_{j_t} where the indices are all at least three and are not necessarily distinct. In much the same manner as Lemma 2 and Lemma 3, an irregular k -coloring of G corresponds to a subgraph H of $\mathcal{D}_3(k)$ composed of conjugate-avoiding cycles and boundary distinguished conjugate-avoiding paths with the added property that if w and \bar{w} are in H then $w = \bar{w}$. As implied by Lemma 1, if such a k -coloring of G were to exist then $n_2(G) \leq k \binom{k}{2}$. We conjecture that this is nearly optimal.

Conjecture 1. *Let G be a graph with $\Delta(G) \leq 2$ and let k be the unique integer such that*

$$(k-1) \binom{k-1}{2} + 1 \leq n_2(G) \leq k \binom{k}{2}.$$

Then $\chi_{irr}(G) \leq k+1$.

Also, while this paper was under review, we received notice that Theorems 1 and 3 were obtained independently in [3]. While [3] did not utilize structured deBruijn sequences, the interested reader may wish examine the approach there, as it provides an interesting contrast to the techniques employed in this paper.

Acknowledgement: The authors would like to thank the referees for their thoughtful comments, which improved both the exposition and clarity of this paper.

REFERENCES

- [1] M. Albertson, D. Boutin, Distinguishing geometric graphs, *J. Graph Theory* **53** (2006), no. 2, 135–150.
- [2] M. Albertson, K. Collins, Symmetry breaking in graphs. *Electron. J. Combin.* **3** (1996) R18.
- [3] M. Anderson, C. Barrientos, R. Brigham, J. Carrington, M. Kronman, R. Vitray and J. Yellen, The Irregular Chromatic Number of Some Graph Classes, to appear in *Bull. Inst. Combin. Appl.*
- [4] J. Anderson, K. Fox, G. Niblo, A fast algorithm for the construction of universal footprinting templates in DNA. *J. Math. Biol.* 52 (2006), no. 3, 307-342.
- [5] K. Appel, W. Haken. Every planar map is four colorable. *Bull. Amer. Math. Soc.* **82** (1976), no. 5, 711-712.
- [6] P. Balister, B. Bollobas, R. Schelp, Vertex distinguishing colorings of graphs with $\Delta(G) = 2$. *Discrete Math.* 252 (2002), no. 1-3, 17-29.
- [7] A. Burriss, R. Schelp. Vertex-distinguishing proper edge colorings. *J. Graph Theory* **26** (1997) no 2, 73-82.
- [8] G. Chartrand, H. Escudro, F. Okamoto, P. Zhang. Detectable colorings of graphs, *Util. Math.* To appear.
- [9] G. Chartrand and L. Lesniak. *Graphs and Digraphs*, Fourth Edition. Chapman & Hall/ CRC. (2005).
- [10] G. Chartrand, M. S. Jacobson, J. Lehel, O. Oellerman, S. Ruiz and F. Saba, Irregular Networks, *Congr. Num.*, **64** (1988) 197-210.
- [11] K. Collins, A. Trenk. The distinguishing chromatic number. *Electron. J. Combin.* **13** (2006), no. 1, R16.
- [12] N. G. deBruijn. A combinatorial problem, *Koninklijke Nederlandse Akademie v. Wetenschappen*, **49** (1946) 758-764.
- [13] F Harary, R.A. Melter. On the metric dimension of a graph. *ARS Combin.* **2** (1976) 191-195.
- [14] W. Imrich, S. Klavžar, Distinguishing Cartesian powers of graphs, *J. Graph Theory* **53** (2006), no. 3, 250-260.
- [15] F. Okamoto, M. Radcliffe, P. Zhang. On the Irregular Chromatic Number of a Graph. *Congr. Num.*, **181** (2006) 129-150.
- [16] N. Robertson, D. Sanders, P. Seymour, R. Thomas. A new proof of the four-colour theorem. *Electron. Res. Announc. Amer. Math. Soc.* **2** (1996), no. 1, 17–25.
- [17] M. Radcliffe, P. Zhang. Irregular Colorings of Graphs, *Bull. Inst. Combin. Appl.*, **49** (2007) 41-59.
- [18] P. Slater. Leaves of Trees. *Congr. Num.*, **14** (1975) 549-559.