Assignment # 7 Math 2411

$\boxed{8.8}$

(4) $\displaystyle\int_{-1}^{\infty} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \int_{-1}^{b} \frac{x}{1+x^2}\,dx$

u-sub
$u = 1+x^2$

$\displaystyle = \lim_{b\to\infty} \frac{1}{2}\Big[\ln|1+x^2|\Big]_{-1}^{b}$

$\displaystyle = \frac{1}{2}\lim_{b\to\infty}\Big[\ln|1+b^2| - \ln|1+(-1)^2|\Big]$

$\displaystyle = \frac{1}{2}\Big[\infty - \ln(2)\Big]$

$= \infty$ diverges

(6) $\displaystyle\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$

u-sub
$u = -x^2$

$\displaystyle = \lim_{b\to\infty}\Big[-\frac{1}{2}e^{-x^2}\Big]_0^b$

$\displaystyle = -\frac{1}{2}\lim_{b\to\infty}\Big[e^{-b^2} - e^{-0^2}\Big]$

$\displaystyle = -\frac{1}{2}\Big[0 - 1\Big]$

$\displaystyle = \frac{1}{2}$

(10) $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9} = \lim_{a\to-\infty}\int_a^3 \frac{1}{x^2+3^2}\,dx$

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$\displaystyle = \lim_{a\to-\infty}\Big[\frac{1}{3}\tan^{-1}\Big(\frac{x}{3}\Big)\Big]_a^3$

(10) continued...

$\displaystyle = \frac{1}{3}\lim_{a\to-\infty}\Big[\tan^{-1}\Big(\frac{3}{3}\Big) - \tan^{-1}\Big(\frac{a}{3}\Big)\Big]$

$\displaystyle = \frac{1}{3}\Big[\tan^{-1}(1) - \lim_{a\to-\infty}\tan^{-1}\Big(\frac{a}{3}\Big)\Big]$

$\displaystyle = \frac{1}{3}\Big[\frac{\pi}{4} - \Big(-\frac{\pi}{2}\Big)\Big]$

as $a\to\infty$,
$\frac{a}{3}\to\infty$
$\tan^{-1}\big(\frac{a}{3}\big)\to$

$\displaystyle = \frac{1}{3}\Big[\frac{3\pi}{4}\Big]$

$\displaystyle = \frac{\pi}{4}$

(14)
$\displaystyle\int_{-\infty}^{\infty}\frac{x}{\sqrt{x^2+2}}\,dx = \int_{-\infty}^{0}\frac{x}{\sqrt{x^2+2}}\,dx + \int_0^{\infty}\frac{x}{\sqrt{x^2+2}}\,dx$

Lets find the indefinite integral first:

$\displaystyle\int\frac{x}{\sqrt{x^2+2}}\,dx = \int\frac{1}{\sqrt{u}}\cdot\frac{du}{2}$

$u = x^2+2$
$\frac{du}{dx} = 2x$
$\frac{du}{2} = x\,dx$

$\displaystyle = \frac{1}{2}\int u^{-1/2}\,du$

$\displaystyle = \frac{1}{2}\Big[2u^{1/2}\Big] + C$

$= \sqrt{x^2+2} + C$

$\displaystyle\int_0^{\infty}\frac{x}{\sqrt{x^2+2}}\,dx = \lim_{b\to\infty}\int_0^b\frac{x}{\sqrt{x^2+2}}\,dx$

$\displaystyle = \lim_{b\to\infty}\Big[\sqrt{x^2+2}\Big]_0^b$

$\displaystyle = \lim_{b\to\infty}\Big[\sqrt{b^2+2} - \sqrt{2}\Big]$

$= \infty$

Integral Diverges

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$$\int_{-\infty}^{\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt = \int_{-\infty}^{0} \frac{e^{-t}}{1+e^{-2t}}\,dt + \int_{0}^{\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt$$

Lets find the indefinite integral 1st:

$$\int \frac{e^{-t}}{1+e^{-2t}}\,dt = \int \frac{e^{-t}}{1+(e^{-t})^2}\,dt$$

Let $u = e^{-t}$

$\frac{du}{dt} = -e^{-t}$

$-du = e^{-t}\,dt$

$$= \int \frac{-du}{1+u^2}$$

$$= -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1}(u) + C$$

$$\int_{-\infty}^{0} \frac{e^{-t}}{1+e^{-2t}}\,dt = \lim_{a\to-\infty} \int_{a}^{0} \frac{e^{-t}}{1+e^{-2t}}\,dt$$

$$= \lim_{a\to-\infty} \Big[-\tan^{-1}(e^{-t})\Big]_a^0$$

$$= \lim_{a\to-\infty} \Big[-\tan^{-1}(e^{0}) + \tan^{-1}(e^{-a})\Big]$$

$$= -\tan^{-1}(1) + \frac{\pi}{2}$$

$$= -\frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\int_{0}^{\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt = \lim_{b\to\infty} \int_{0}^{b} \frac{e^{-t}}{1+e^{-2t}}\,dt$$

$$= \lim_{b\to\infty} \Big[-\tan^{-1}(e^{-t})\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[-\tan^{-1}(e^{-b}) + \tan^{-1}(e^{0})\Big]$$

$$= -\tan^{-1}(0) + \tan^{-1}(1)$$

16 continued...

$$= -0 + \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

So
$$\int_{-\infty}^{\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

18

$$\int_0^8 \frac{dx}{\sqrt[3]{x}} = \lim_{a\to0^+} \int_a^8 x^{-1/3}\,dx$$

$$= \lim_{a\to0^+} \Big[\frac{3}{2} x^{2/3}\Big]_a^8$$

$$= \lim_{a\to0^+} \Big[\frac{3}{2}\cdot 8^{2/3} - \frac{3}{2} a^{2/3}\Big]$$

$$= \frac{3}{2}\cdot 4 - \frac{3}{2}\cdot 0$$

$$= 6$$

26 The function is discontinuous at $x = 0$.

$$\int_{-2}^{2} \frac{dx}{x^2} = \int_{-2}^{0} \frac{dx}{x^2} + \int_{0}^{2} \frac{dx}{x^2}$$

$$= \lim_{b\to0^-} \int_{-2}^{b} \frac{dx}{x^2} + \lim_{a\to0^+} \int_{a}^{2} \frac{dx}{x^2}$$

$$= \lim_{b\to0^-} \Big[-\frac{1}{x}\Big]_{-2}^{b} + \lim_{a\to0^+} \Big[-\frac{1}{x}\Big]_a^2$$

$$= \lim_{b\to0^-} \Big[-\frac{1}{b} + \frac{1}{-2}\Big] + \lim_{a\to0^+} \Big[-\frac{1}{2} + \frac{1}{a}\Big]$$

$$= \left(\infty - \frac{1}{2}\right) + \left(-\frac{1}{2} + \infty\right)$$

$= \infty$ diverges

8.8

(36) $\int_0^\infty x e^{-3x}\,dx = \lim_{b\to\infty}\int_0^b x e^{-3x}\,dx$

TI-89 $\curvearrowright = \frac{1}{9}$

$\int_0^\infty x e^{-3x}\,dx \overset{\text{TI-89}}{=} \frac{1}{9}$

9.1

(10) $\frac{dy}{dx} + 2xy = x$

$p(x) = 2x$

Step 1) $\mu(x) = e^{\int p(x)dx} = e^{\int 2x dx} = e^{x^2}$

Step 2) $e^{x^2}\left(\frac{dy}{dx} + 2xy\right) = xe^{x^2}$

$e^{x^2}\cdot\frac{dy}{dx} + e^{x^2}2xy = xe^{x^2}$

Step 3) $\frac{d}{dx}[e^{x^2}y] = xe^{x^2}$

Step 4) $e^{x^2}y = \int xe^{x^2}dx$

$e^{x^2}y = \frac{1}{2}e^{x^2} + C$

$e^{-x^2}(e^{x^2}y) = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right)$

$y = \frac{1}{2} + Ce^{-x^2}$

(14) $\frac{dy}{dx} + y = -\frac{1}{1-e^x}$

$p(x) = 1$

(Step 1) $\mu(x) = e^{\int p(x)dx} = e^{\int 1 dx} = e^x$

(Step 2) $e^x\left(\frac{dy}{dx} + y\right) = e^x\left(-\frac{1}{1-e^x}\right)$

(14) continued,,,

$e^x\frac{dy}{dx} + e^x y = -\frac{e^x}{1-e^x}$

(Step 3) $\frac{d}{dx}[e^x y] = -\frac{e^x}{1-e^x}$

$e^x y = \int \frac{-e^x}{1-e^x}dx$

$u = 1-e^x$
$\frac{du}{dx} = -e^x$
$du = -e^x dx$

$e^x y = \int \frac{du}{u}$

$u = 1-e^x$
$du = -e^x dx$

$e^x y = \ln|u| + C$

$e^x y = \ln|1-e^x| + C$

$y = e^{-x}\ln|1-e^x| + Ce^{-x}$

(16) $\frac{dy}{dx} = 2(1+y^2)x$

$\frac{dy}{1+y^2} = 2x\,dx$

$\int\frac{dy}{1+y^2} = \int 2x\,dx$

$\tan^{-1}(y) = x^2 + C$

$y = \tan(x^2 + C)$

(20) $\frac{dy}{dx} = -xy$

$\frac{dy}{y} = -x\,dx$

$\int\frac{dy}{y} = \int -x\,dx$

$\ln|y| = -\frac{x^2}{2} + C$

$e^{\ln|y|} = e^{-\frac{x^2}{2}+C}$

9.1

20 continued...

$$|y| = e^{-\frac{x^2}{2}} e^{c}$$

$$|y| = e^{c} e^{-\frac{x^2}{2}}$$

$$y = \pm e^{c} e^{-\frac{x^2}{2}}$$

$$y = K e^{-\frac{x^2}{2}}$$

24 $$y - \frac{dy}{dx}\sec x = 0$$

$$-\frac{dy}{dx}\sec x = -y$$

$$\frac{dy}{dx}\sec x = y$$

$$\frac{dy}{y} = \frac{dx}{\sec x}$$

$$\frac{dy}{y} = \cos x\, dx$$

$$\int \frac{dy}{y} = \int \cos x\, dx$$

$$\ln|y| = \sin x + C$$

$$e^{\ln|y|} = e^{\sin x + C}$$

$$|y| = e^{c} e^{\sin x}$$

$$y = \pm e^{c} e^{\sin x}$$

$$y = K e^{\sin x}$$

28 $$\underbrace{\frac{dy}{dx} + y}_{p(x)=1} = 2 , \quad y(0) = 1$$

(Step 1) $\mu(x) = e^{\int p(x)dx} = e^{\int 1 dx} = e^{x}$

(Step 2) $e^{x}\left(\frac{dy}{dx} + y\right) = 2e^{x}$

$$e^{x}\frac{dy}{dx} + e^{x}y = 2e^{x}$$

(Step 3) $\frac{d}{dx}\left[e^{x} y\right] = 2e^{x}$

(Step 4) $e^{x} y = \int 2e^{x} dx$

$$e^{x}y = 2e^{x} + C$$

$$e^{-x}(e^{x}y) = e^{-x}(2e^{x} + C)$$

$$y = 2 + Ce^{-x}$$

General Solution

Now we apply the initial condition and solve for C,

$y(0) = 1 \Rightarrow$

$$1 = 2 + Ce^{-0}$$

$$1 = 2 + C$$

$$-1 = C$$

So the solution to the IVP is:

$$y = 2 - e^{-x}$$