Assignment #5

8.2

(10)

$u = \ln x$	$dv = x^{\frac{1}{2}} dx$
$du = \frac{1}{x} dx$	$v = \frac{2}{3} x^{3/2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int (\ln x)(x^{\frac{1}{2}})dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\left(\frac{1}{x}\right)dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left[\frac{2}{3}x^{3/2}\right] + C$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

(16)

$u = \cos^{-1}(2x)$	$dv = dx$
$du = -\frac{2}{\sqrt{1-(2x)^2}}$	$v = x$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^{-1}(2x)dx$$

$$= x\cos^{-1}(2x) - \int -\frac{2x}{\sqrt{1-4x^2}}dx$$

$$= x\cos^{-1}(2x) + 2\int \frac{x}{\sqrt{1-4x^2}}dx$$

The last integral is u-sub. Lets use w instead of u.

(16) continued...

Let $w = 1-4x^2$

$$\frac{dw}{dx} = -8x$$

$$\frac{dw}{-8} = x\,dx$$

So $\int \frac{x}{\sqrt{1-4x^2}}dx = \int \frac{1}{\sqrt{w}} \cdot \frac{dw}{-8}$

$$= -\frac{1}{8}\int w^{-1/2}dw$$

$$= -\frac{1}{8}\left[2w^{1/2}\right] + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Thus

$$\int \cos^{-1}(2x)\,dx$$

$$= x\cos^{-1}(2x) + 2\left(-\frac{1}{4}\sqrt{1-4x^2}\right) + C$$

$$= x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + C$$

(18)

$u = \tan^{-1}x$	$dv = x\,dx$
$du = \frac{1}{1+x^2}dx$	$v = \frac{x^2}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\tan^{-1}x\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

The last integral can be solved by long division.

(18) continued...

$$
\begin{array}{r}
1 \\
x^2+1 \,\big)\, \overline{x^2+0x+0} \\
-x^2 -1 \\
\hline
-1
\end{array}
$$

-1 ← Remainder

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

So $$\int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \int dx - \int \frac{1}{x^2+1}\,dx$$

$$= x - \tan^{-1}x + C$$

Thus

$$\int x\tan^{-1}x\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$$

$$= \left(\frac{x^2+1}{2}\right)\tan^{-1}x - \frac{x}{2} + C$$

(22)

$u = \sin(5\theta)$	$dv = e^{-3\theta}$
$du = 5\cos(5\theta)\,d\theta$	$v = -\frac{1}{3}e^{-3\theta}$

$$\int e^{-3\theta}\sin(5\theta)\,d\theta$$

$$= -\frac{1}{3}e^{-3\theta}\sin(5\theta) - \int -\frac{1}{3}e^{-3\theta}\cdot 5\cos(5\theta)\,d\theta$$

$$= -\frac{1}{3}e^{-3\theta}\sin(5\theta) + \frac{5}{3}\int e^{-3\theta}\cos(5\theta)\,d\theta$$

Parts again on this integral

(22) continued...

$u = \cos 5\theta$	$dv = e^{-3\theta}$
$du = -5\sin 5\theta$	$v = -\frac{1}{3}e^{-3\theta}$

$$\int e^{-3\theta}\cos(5\theta)\,d\theta$$

$$= -\frac{1}{3}e^{-3\theta}\cos(5\theta) - \int -\frac{1}{3}e^{-3\theta}\cdot -5\sin(5\theta)\,d\theta$$

$$= -\frac{1}{3}e^{-3\theta}\cos(5\theta) - \frac{5}{3}\int e^{-3\theta}\sin(5\theta)\,d\theta$$

The unknown integral, call it I^{*}, has repeated and is on both sides of the equation! So we solve for the unknown integral.

$$I^{*} = -\frac{1}{3}e^{-3\theta}\sin(5\theta) + \frac{5}{3}\left[-\frac{1}{3}e^{-3\theta}\cos(5\theta) - \frac{5}{3}I^{*}\right]$$

$$I^{*} = -\frac{1}{3}e^{-3\theta}\sin(5\theta) - \frac{5}{9}e^{-3\theta}\cos(5\theta) - \frac{25}{9}I^{*}$$

$$+\frac{25}{9}I^{*} \qquad\qquad +\frac{25}{9}I^{*}$$

$$\frac{34}{9}I^{*} = -\frac{1}{3}e^{-3\theta}\sin(5\theta) - \frac{5}{9}e^{-3\theta}\cos(5\theta)$$

$$I^{*} = -\frac{3}{34}e^{-3\theta}\sin(5\theta) - \frac{5}{34}e^{-3\theta}\cos(5\theta)$$

$$= -\frac{e^{-3\theta}}{34}\left(3\sin(5\theta) + 5\cos(5\theta)\right)$$

(30)

$u = x$	$dv = e^{-5x}dx$
$du = dx$	$v = -\frac{1}{5}e^{-5x}$

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

$$\int_0^1 x e^{-5x}dx$$

$$= \left[-\frac{1}{5}xe^{-5x}\right]_0^1 - \int_0^1 -\frac{1}{5}e^{-5x}dx$$

$$= \left[-\frac{1}{5}(1)e^{-5} - 0\right] + \frac{1}{5}\int_0^1 e^{-5x}dx$$

$$= -\frac{1}{5}e^{-5} + \frac{1}{5}\left[-\frac{1}{5}e^{-5x}\right]_0^1$$

$$= -\frac{e^{-5}}{5} - \frac{1}{25}\left[e^{-5} - e^{0}\right]$$

$$= -\frac{e^{-5}}{5} - \frac{e^{-5}}{25} + \frac{1}{25}$$

$$= -\frac{6e^{-5}}{25} + \frac{1}{25}$$

$$= \frac{1-6e^{-5}}{25}$$

(48)

ⓐ

$u = x^2$	$dv = \frac{x}{\sqrt{x^2+1}}$
$du = 2x\,dx$	$v = \sqrt{x^2+1}$

$$\int x^2\left(\frac{x}{\sqrt{x^2+1}}\right)dx$$

$$= x^2\sqrt{x^2+1} - \int 2x\sqrt{x^2+1}\,dx$$

Let $w = x^2+1$

$dw = 2x\,dx$

(48) ⓐ continued...

$$= x^2\sqrt{x^2+1} - \int w^{1/2}dw$$

$$= x^2\sqrt{x^2+1} - \frac{2}{3}w^{3/2} + C$$

$$= x^2\sqrt{x^2+1} - \frac{2}{3}(x^2+1)^{3/2} + C$$

ⓑ Let $u = (x^2+1)^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$$

$$du = \frac{x}{\sqrt{x^2+1}}dx$$

but we still need x^2:

$$u^2 = x^2+1 \Rightarrow x^2 = u^2-1$$

So we now have

$$\int x^2\left(\frac{x}{\sqrt{x^2+1}}\right)dx$$

$$= \int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u + C$$

Evaluating the definite integral yields:

$$\int_0^1 x^2\left(\frac{x}{\sqrt{x^2+1}}\right)dx = \int_1^{\sqrt{2}} (u^2-1)\,du$$

$$= \left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}}$$

$$= \left(\frac{2^{3/2}}{3} - \sqrt{2}\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{2\sqrt{2}}{3} - \sqrt{2} + \frac{2}{3}$$

$$= \frac{2}{3} - \frac{\sqrt{2}}{3}$$

(54) By Tabular Integration

$$\begin{array}{lll} t & \sin(k\omega t) & \\ 1 & -\frac{1}{k\omega}\cos(k\omega t) & \xrightarrow{+} -\frac{t}{k\omega}\cos(k\omega t) \\ 0 & -\frac{1}{k^2\omega^2}\sin(k\omega t) & \xrightarrow{-} +\frac{1}{k^2\omega^2}\sin(k\omega t) \end{array}$$

So $\displaystyle\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)$

$$= \left[-\frac{t}{k\omega}\cos(k\omega t) + \frac{1}{k^2\omega^2}\sin(k\omega t)\right]_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}$$

$$= -\frac{\frac{\pi}{\omega}}{k\omega}\cos(k\pi) + \frac{1}{k^2\omega^2}\sin(k\pi)$$

$$-\left(-\frac{-\frac{\pi}{\omega}}{k\omega}\cos(-k\pi) + \frac{1}{k^2\omega^2}\sin(-k\pi)\right)$$

$$= -\frac{\pi}{k\omega^2}\cos(k\pi) + \frac{1}{k^2\omega^2}\sin(k\pi)$$

$$-\frac{\pi}{k\omega^2}\cos(-k\pi) - \frac{1}{k^2\omega^2}\sin(-k\pi)$$

$$= -\frac{\pi}{k\omega^2}\cos(k\pi) - \frac{1}{k\omega^2}\cos(-k\pi)$$

{since k is an integer, $\sin(k\pi) = 0$}

$\boxed{8.3}$

(8) $\displaystyle\int \sin^3 x\cos^3 x\,dx$

$$= \int \sin^3 x\cos^2 x\cos x\,dx$$

$$= \int \sin^3 x(1-\sin^2 x)\cos x\,dx$$

$$\left\{\begin{array}{l}\text{Let } u = \sin x \\ du = \cos x\,dx\end{array}\right\}$$

$$= \int u^3(1-u^2)\,du$$

$$= \int (u^3 - u^5)\,du$$

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

(10) $\displaystyle\int \sin^3 x\cos^2 x\,dx$

$$= \int \sin^2 x\cos^2 x\sin x\,dx$$

$$= \int (1-\cos^2 x)\cos^2 x\sin x\,dx$$

$$\left\{\begin{array}{l}\text{Let } u = \cos x \\ du = -\sin x\,dx\end{array}\right\}$$

$$= -\int (1-u^2)u^2\,du$$

$$= -\int (u^2 - u^4)\,du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

$\boxed{8.3}$

(18) Lets find the indefinite integral first. We can use the identities:

$$\sin^2\theta = \frac{1}{2}\left[1 - \cos(2\theta)\right]$$

$$\Rightarrow \sin^2\left(\frac{x}{2}\right) = \frac{1}{2}\left[1 - \cos(x)\right]$$

and $\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$

$$\Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1}{2}\left[1 + \cos(x)\right]$$

Thus

$$\int \sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) dx$$

$$= \int \left[\frac{1}{2}(1 - \cos x)\cdot\frac{1}{2}(1+\cos x)\right] dx$$

$$= \frac{1}{4}\int (1 - \cos^2 x)\, dx$$

$$= \frac{1}{4}\int 1\, dx - \frac{1}{4}\int \cos^2 x\, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}\left[1 + \cos(2x)\right] dx$$

$$= \frac{1}{4}x - \frac{1}{8}\int dx - \frac{1}{8}\int \cos(2x)\,dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\left(\frac{1}{2}\sin(2x)\right) + C$$

$$= \frac{1}{8}x - \frac{1}{16}\sin(2x) + C$$

$$\int_0^{\pi/2} \sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) dx$$

$$= \left[\frac{x}{8} - \frac{1}{16}\sin(2x)\right]_0^{\frac{\pi}{2}}$$

(18) continued...

$$= \left(\frac{\pi}{16} - \frac{1}{16}\sin(\pi)\right) - (0 - 0)$$

$$= \frac{\pi}{16} - \frac{1}{16}(0)$$

$$= \frac{\pi}{16}$$

(20)

$$\int \cos^2(5\theta)\,d\theta$$

$$= \int \frac{1}{2}\left[1 + \cos(10\theta)\right] d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos(10\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\left[\frac{1}{10}\sin(10\theta)\right] + C$$

$$= \frac{\theta}{2} + \frac{1}{20}\sin(10\theta) + C$$

So

$$\int_{-\pi}^{\pi} \cos^2(5\theta)\,d\theta$$

$$= \left[\frac{\theta}{2} + \frac{1}{20}\sin(10\theta)\right]_{-\pi}^{\pi}$$

$$= \left(\frac{\pi}{2} + \frac{1}{20}\sin(10\pi)\right) - \left(\frac{-\pi}{2} + \frac{1}{20}\sin(-10\pi)\right)$$

$$= \left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

30

$$\int \tan^5 x \sec^4 x \, dx$$

$$= \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^5 x \left(\tan^2 x + 1\right) \sec^2 x \, dx$$

$$\left\{ \begin{array}{l} \text{Let } u = \tan x \\ \quad du = \sec^2 x \, dx \end{array} \right\}$$

$$= \int u^5 (u^2 + 1) \, du$$

$$= \int (u^7 + u^5) \, du$$

$$= \frac{1}{8} u^8 + \frac{1}{6} u^6 + C$$

or

$$= \frac{1}{8} \tan^8(x) + \frac{1}{6} \tan^6(x) + C$$

34

$$\int \tan^5 \theta \sec \theta \, d\theta$$

$$= \int \tan^4 \theta \sec \theta \tan \theta \, d\theta$$

$$= \int (\tan^2 \theta)^2 \sec \theta \tan \theta \, d\theta$$

$$= \int (\sec^2 \theta - 1)^2 \sec \theta \tan \theta \, d\theta$$

$$= \int (\sec^4 \theta - 2\sec^2 \theta + 1) \sec \theta \tan \theta \, d\theta$$

$$\left\{ \begin{array}{l} \text{Let } u = \sec \theta \\ \quad du = \sec \theta \tan \theta \, d\theta \end{array} \right\}$$

34 continued...

$$= \int (u^4 - 2u^2 + 1) \, du$$

$$= \frac{u^5}{5} - \frac{2}{3} u^3 + u + C$$

$$= \frac{1}{5} \sec^5 \theta - \frac{2}{3} \sec^3 \theta + \sec \theta + C$$

30 can also be done as follows:

$$\int \tan^5 x \sec^4 x \, dx$$

$$= \int \tan^4 x \sec^3 x \sec x \tan x \, dx$$

$$= \int (\tan^2 x)^2 \sec^3 x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x \, dx$$

$$\left\{ \begin{array}{l} \text{Let } u = \sec x \\ \text{then } du = \sec x \tan x \, dx \end{array} \right.$$

$$= \int (u^2 - 1)^2 u^3 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^3 \, du$$

$$= \int (u^7 - 2u^5 + u^3) \, du$$

$$= \frac{u^8}{8} - \frac{2u^6}{6} + \frac{u^4}{4} + C$$

$$= \frac{1}{8} \sec^8 x - \frac{1}{3} \sec^6 x + \frac{1}{4} \sec^4 x + C$$