Solutions to Material for Test #2 – MATH 2411

(#1) If we give you this reduction formula:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx,$$

then evaluate this antiderivative:

$$\int \cos^4(\theta)\,d\theta = ???$$

For $n = 4$, we have

$$\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx.$$

You have $\int \cos^2(\theta)\,d\theta$ on your note sheet.

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left(\frac{x}{2} + \frac{\sin(2x)}{4}\right) + C \quad \text{or} \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left(\frac{x}{2} + \frac{\sin(x)\cos(x)}{2}\right) + C.\end{aligned}$$

(#2) Evaluate:

(a) Integration by parts (IBP).

$$\int_1^e x^4 \ln(x)\,dx = ???$$

We always put the natural log function in the "u" part.

$$\begin{aligned} u &= \ln(x),\ du = \frac{1}{x}dx \\ dv &= x^4,\ v = \frac{x^5}{5}\end{aligned}$$

IBP:

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ &= \frac{1}{5}x^5\ln(x) - \int\left(\frac{x^5}{5}\right)\left(\frac{1}{x}dx\right) = \frac{1}{5}x^5\ln(x) - \frac{1}{5}\int x^4 dx \\ &= \frac{1}{5}x^5\ln(x) - \frac{1}{5}\left(\frac{x^5}{5}\right) + C = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C.\end{aligned}$$

We evaluate the definite integral.

$$\begin{aligned}\int_1^e x^4\ln(x)\,dx &= \frac{1}{5}\left[x^5\ln(x)\right]_1^e - \frac{1}{25}\left[x^5\right]_1^e \\ &= \frac{1}{5}\left(e^5\ln(e) - 1\ln(1)\right) - \frac{1}{25}\left(e^5 - 1\right) \\ &= \frac{1}{5}e^5 - \frac{1}{25}e^5 + \frac{1}{25} = \frac{4e^5+1}{25}.\end{aligned}$$

(b)

$$\int \sin^{-1}(2x)\,dx$$

IBP:

$$\begin{aligned} u &= \sin^{-1}(2x), \ du = \frac{2dx}{\sqrt{1-(2x)^2}} \\ dv &= dx, \ v = x. \end{aligned}$$

The antiderivative is

$$uv - \int v\ du = x\sin^{-1}(2x) - 2\int \frac{x}{\sqrt{1-4x^2}}dx.$$

We have a u-sub.

$$u = 1-4x^2, \ du = -8x\ dx \Rightarrow x\ dx = -\frac{1}{8}du.$$

Thus, we have

$$\begin{aligned} \int \frac{x}{\sqrt{1-4x^2}} \ \Rightarrow \ \int \frac{(-1/8)\,du}{\sqrt{u}} &= -\frac{1}{8}\int u^{-1/2}du = -\frac{1}{8}\left(\frac{u^{1/2}}{(1/2)}\right) + C \\ &= -\frac{1}{4}\sqrt{1-4x^2} + C. \end{aligned}$$

Thus, our IBP antiderivative is

$$\begin{aligned} \int \sin^{-1}(2x)\,dx &= x\sin^{-1}(2x) - 2\left(-\frac{1}{4}\sqrt{1-4x^2}\right) + C \\ &= x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C. \end{aligned}$$

(c)

$$\int x\sec^2(x)\,dx$$

IBP:

$$\begin{aligned} u &= x, \ du = dx \\ dv &= \sec^2(x), \ v = \tan(x). \end{aligned}$$

The antiderivative is

$$\begin{aligned} uv - \int v\ du &= x\tan(x) - \int \tan(x)\,dx \\ &= x\tan(x) - (-\ln|\cos(x)|) + C \\ &= x\tan(x) + \ln|\cos(x)| + C. \end{aligned}$$

(#3) Suppose we have the region trapped beneath $y = \sin(x)$ over $[0, \pi]$, above the x-axis.

We create a solid of revolution whose volume can be evaluated by the Disk Method. Find the volume.

$$\begin{aligned} V &= \pi \int_0^{\pi} y^2 dx = \pi \int_0^{\pi} \sin^2(x)\, dx \\ &= \pi \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_0^{\pi} = \pi \left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} \right) = \frac{\pi^2}{2} \text{ cubic units.} \end{aligned}$$

(#4) Evaluate:

(a)

$$\int \cos^2(3x) \sin^3(3x)\, dx = ???$$

There are an odd number of sine factors, so convert 2 of them to cosines.

$$\begin{aligned} I &= \int \cos^2(3x) \left(1 - \cos^2(3x)\right) \sin(3x)\, dx \\ &= \int \left(\cos^2(3x) - \cos^4(3x)\right) \sin(3x)\, dx \end{aligned}$$

$u = \cos(3x)\,,\ du = -3\sin(3x)\, dx \Rightarrow \sin(3x)\, dx = (-1/3)\, du.$

$$-\frac{1}{3} \int \left(u^2 - u^4\right) du = -\frac{1}{3} \left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C.$$

Our antiderivative is

$$-\frac{1}{3} \left(\frac{\cos^3(3x)}{3} - \frac{\cos^5(3x)}{5} \right) + C.$$

(b)

$$\int_0^{\pi/4} \tan(\theta) \sec^4(\theta)\, d\theta = ???$$

There are an even number of secant factors, so convert 2 of them to tangents.

$$\begin{aligned} I &= \int_0^{\pi/4} \tan(\theta) \left(1 + \tan^2(\theta)\right) \sec^2(\theta)\, d\theta \\ &= \int_0^{\pi/4} \left(\tan(\theta) + \tan^3(\theta)\right) \sec^2(\theta)\, d\theta \end{aligned}$$

$u = \tan(\theta)\,,\ du = \sec^2(\theta)\, d\theta.$
When $\theta = 0,\ u = 0.$
When $\theta = \pi/4,\ u = 1.$

$$I = \int_0^1 \left(u + u^3\right) du = \left[\frac{u^2}{2} + \frac{u^4}{4} \right]_0^1 = \frac{3}{4}.$$

(#5) Evaluate:

(a) Trig. sub.

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = ???$$

$x = \sin(\theta)\,,\ dx = \cos(\theta)\, d\theta$
Also, by Pythagoras, we have

$$\sin(\theta) = \frac{x}{1} = \frac{opp}{hyp}$$

and adjacent side is $\sqrt{1-x^2}$. Thus, we have

$$\cos(\theta) = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

The new integral is

$$\int \frac{\cos(\theta)\, d\theta}{\sin^2(\theta)\cos(\theta)} = \int \frac{d\theta}{\sin^2(\theta)} = \int \csc^2(\theta)\, d\theta.$$

We recall that

$$\int \sec^2(\theta)\, d\theta = \tan(\theta) + C$$

and that if we substitute in the co-functions (complementary functions), we simply take the negative of the result.

$$\int \csc^2(\theta)\, d\theta = -\cot(\theta) + C.$$

Remember that we still need an expression in terms of x.

$$\cot(\theta) = \frac{adj}{opp} = \frac{\sqrt{1-x^2}}{x}$$

Thus, the final antiderivative is

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + C.$$

Try checking it with the Quotient Rule.

(b) Hint: You'll need $\int \sec(\theta)\, d\theta$.

$$\int \frac{x^2}{(x^2+1)^{3/2}} dx = ???$$

$x = \tan(\theta)\,,\ dx = \sec^2(\theta)\, d\theta.$
Also, by Pythagoras, we have

$$x^2 + 1 = \tan^2(\theta) + 1 = \sec^2(\theta)\,.$$

Our new integral is

$$\begin{aligned}
\int \frac{\tan^2(\theta) * \sec^2(\theta)\, d\theta}{\left(\sec^2(\theta)\right)^{3/2}} &= \int \frac{\tan^2(\theta) * \sec^2(\theta)\, d\theta}{\sec^3(\theta)} = \int \frac{\tan^2(\theta)\, d\theta}{\sec(\theta)} \\
&= \int \frac{\sec^2(\theta) - 1}{\sec(\theta)} d\theta = \int \left(\sec(\theta) - \frac{1}{\sec(\theta)}\right) d\theta \\
&= \int \left(\sec(\theta) - \cos(\theta)\right) d\theta = \ln|\sec(\theta) + \tan(\theta)| - \sin(\theta) + C.
\end{aligned}$$

Our right triangle has:

$$\tan(\theta) = \frac{opp}{adj} = \frac{x}{1} \Rightarrow hyp = \sqrt{1 + x^2}.$$

Thus, we have

$$\begin{aligned}
\sec(\theta) &= \frac{hyp}{adj} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2} \\
\sin(\theta) &= \frac{opp}{hyp} = \frac{x}{\sqrt{1+x^2}}.
\end{aligned}$$

Our antiderivative is

$$\int \frac{x^2}{(x^2+1)^{3/2}} dx = \ln\left|\sqrt{1+x^2} + x\right| - \frac{x}{\sqrt{1+x^2}} + C.$$

(#6) Evaluate:

$$\int \frac{11x + 17}{2x^2 + 7x - 4} dx = ???$$

We factor the denominator. Partial fractions.

$$\int \frac{11x+17}{(2x-1)(x+4)} dx = \int \left(\frac{A}{2x-1} + \frac{B}{x+4}\right) dx$$

Find A and B.

$$\frac{A}{2x-1} + \frac{B}{x+4} = \frac{A(x+4) + B(2x-1)}{(2x-1)(x+4)} = \frac{(A+2B)x + (4A - B)}{(2x-1)(x+4)}$$

The numerators must match exactly.

$$\begin{aligned}
A + 2B &= 11 \\
4A - B &= 17
\end{aligned}$$

Multiply the bottom equation by 2 and then add.

$$\begin{aligned}
A + 2B &= 11 \\
8A - 2B &= 34 \\
9A &= 45 \Rightarrow A = 5,\ B = 3.
\end{aligned}$$

Now integrate.

$$\int \left(\frac{5}{2x-1} + \frac{3}{x+4}\right) dx = \frac{5}{2}\ln|2x-1| + 3\ln|x+4| + C.$$

(#7) Find the partial fraction decomposition for:

$$\frac{3x^2+3x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Repeated linear factors only require a constant coefficient.
A factor such as (x^2+4) requires a numerator of the form $Ax+B$.

$$\begin{aligned} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} &= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2} \\ &= \frac{A(x^2+2x+1) + B(x^2+x) + Cx}{x(x+1)^2} \\ &= \frac{(A+B)x^2 + (2A+B+C)x + A}{x(x+1)^2}. \end{aligned}$$

The numerators must match exactly.

$$\begin{aligned} A+B &= 3 \\ 2A+B+C &= 3 \\ A &= 1 \end{aligned}$$

Since $A=1$, the top equation gives us $B=2$. The second equation gives us $C=-1$.

$$\frac{3x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{2}{x+1} - \frac{1}{(x+1)^2}.$$

These are easy to integrate.

(#8) Evaluate. If the integral diverges, then show why this is true.

(a) Vertical asymptote at $x=0$.

$$\int_0^1 \frac{1}{\sqrt[3]{x}}dx = ???$$

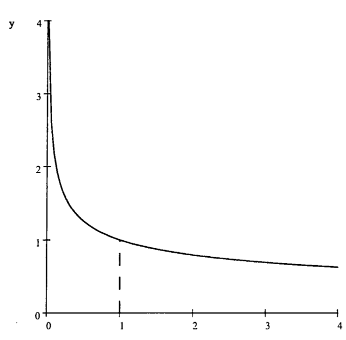
Power rule.

$$\int x^{-1/3}dx = \frac{x^{2/3}}{(2/3)} + C = \frac{3}{2}x^{3/2} + C.$$

Can we evaluate this as $x \to 0^+$? Yes.

$$\frac{3}{2}\left[x^{3/2}\right]_0^1 = \frac{3}{2}.$$

The trapped area is finite.



The vertical "tail" pinches down fast enough and the integral converges.

(b) This was a homework problem.

$$\int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} dt = ???$$

$u = e^{-t}$, $du = -e^{-t}dt \Rightarrow e^{-t}dt = -du.$

$$\int \frac{e^{-t}}{1+e^{-2t}} dt \Rightarrow -\int \frac{du}{1+u^2} = -\tan^{-1}(u) + C.$$

So our definite integral is

$$-\left[\tan^{-1}\left(e^{-t}\right)\right]_0^{+\infty} = -\left(\lim_{b\to+\infty} \tan^{-1}\left(e^{-b}\right) - \tan^{-1}\left(e^{-0}\right)\right).$$

Since $\tan^{-1}$ is continuous, the limit symbol can walk through.

$$I = -\left(\tan^{-1}\left(\lim_{b\to+\infty} e^{-b}\right) - \tan^{-1}(1)\right) = -\left(\tan^{-1}(0) - \frac{\pi}{4}\right) = \frac{\pi}{4}.$$

The integral converges and the area is finite.

(c)

$$\int_{-\infty}^{+\infty} x\ dx = ???$$

Since both limits of integration are improper, we must choose a convenient value of x somewhere in between and break up the integral into two pieces. Let's use $x = 0$.

$$\int_{-\infty}^{+\infty} x\ dx = \int_{-\infty}^{0} x\ dx + \int_0^{+\infty} x\ dx.$$

Consider the second integral.

$$\int_0^{+\infty} x\ dx = \left[\frac{x^2}{2}\right]_0^{+\infty}$$

We see that the expression grows without bound when we attempt to substitute $b = +\infty$.

$$\lim_{b\to+\infty} \frac{b^2}{2} = +\infty.$$

It turns out that both integrals diverge and thus, the combined integral also DIVERGES. Yes, the two opposing trapped areas are symmetric about the origin (odd function), but each region is infinite. Thus, we must declare the integral DIVERGENT.

(#9) Separable. Solve the IVP.

$$y' = y^2 * \sqrt{x},\ y(1) = 3.$$

Separable.

$$\frac{dy}{dx} = y^2 * \sqrt{x} \Rightarrow \int \frac{dy}{y^2} = \int x^{1/2} dx$$

$$-\frac{1}{y} = \frac{2}{3} x^{3/2} + C.$$

When $x = 1$, we have $y = 3$.

$$-\frac{1}{3} = \frac{2}{3}\left(1^{3/2}\right) + C \Rightarrow C = -1.$$

The unique solution is

$$-\frac{1}{y} = \frac{2}{3}x^{3/2} - 1.$$

If the problem specifically asks for an explicit solution for $y(x)$, then we have a bit more work to do.

$$\begin{aligned} \frac{1}{y} &= 1 - \frac{2}{3}x^{3/2} \\ y &= \frac{1}{1 - \frac{2}{3}x^{3/2}} = \frac{3}{3 - 2x^{3/2}}. \end{aligned}$$

Multiply top and bottom by 3.

(#10) Integrating factor. Solve for the general solution (one constant of integration).

$$y' + 3y = x^2$$

Hint: Tabular integration will help here. We have $p(x) = 3$ and $q(x) = x^2$.

The integrating factor is

$$\mu = e^{\int p(x)dx} = e^{\int 3dx} = e^{3x}.$$

Multiply through. The result is a Product Rule on the left side.

$$\begin{aligned} e^{3x}y' + 3e^{3x}y &= x^2e^{3x} \\ \left[e^{3x}y\right]' &= x^2e^{3x} \end{aligned}$$

Integrate.

$$\begin{aligned} \int \left[e^{3x}y\right]' dx &= \int x^2e^{3x}dx \\ e^{3x}y &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C \end{aligned}$$

Divide through by e^{3x}.

$$y = \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27} + Ce^{-3x}.$$

(#11) Complete Problem #7 on p. 608. [Radon-222]

The exponential decay equation is

$$y(t) = y_0e^{-kt},$$

where k is some positive constant.

The initial amount of Radon atoms is $5 * 10^7$ atoms.

In 3.83 days, the number of atoms is halved.

$$\frac{1}{2}y_0 = y_0 e^{-k(3.83)} \Rightarrow \frac{1}{2} = e^{-3.83k}$$

Take the natural log of both sides.

$$\ln\left(\frac{1}{2}\right) = -3.83k \Rightarrow -\ln(2) = -3.83k \Rightarrow k = \frac{\ln(2)}{3.83}.$$

(a) The initial value problem was

$$y' = -ky, \ y(0) = 5 * 10^7.$$

(b) The formula for $y(t)$ is

$$y(t) = \left(5 * 10^7\right) e^{-(\ln(2)/3.83)t} = \left(5 * 10^7\right)\left(\frac{1}{2}\right)^{t/3.83}.$$

Every 3.83 days, we gain an additional factor of $(1/2)$.

(c) After 30 days, we will have

$$\left(5 * 10^7\right)\left(\frac{1}{2}\right)^{30/3.83} \doteq 2.2 * 10^5 \text{ atoms.}$$

That's about 7.8 half-lives.

(d) If 90% decays, then we only have 10% remaining.

$$\begin{aligned}
(0.1)\left(5 * 10^7\right) &= \left(5 * 10^7\right)\left(\frac{1}{2}\right)^{t/3.83} \\
0.1 &= \left(\frac{1}{2}\right)^{t/3.83} \\
\ln(0.1) &= \left(\frac{t}{3.83}\right)\ln(0.5) \Rightarrow t = 3.83 * \frac{\ln(0.1)}{\ln(0.5)}
\end{aligned}$$

It takes approximately $t = 12.7$ days.

(#12) (a) If *y*(*t*) is the amount of the dissolved salt (in lbs) then

$$\frac{dy}{dt} = \text{RATE IN} - \text{RATE OUT} \quad \text{where}$$

RATE IN = (Concentration In)(Flow Rate In)

RATE OUT = (Concentration Out)(Flow Rate Out)

↑ [lb/min] ↑ [lb/gal] ↑ [gal/min]

$$\frac{dy}{dt} = \text{RATE IN} - \text{RATE OUT}$$

$$= (2 \text{ lb/gal})(3 \text{ gal/min}) - \left(\frac{y}{50} \text{ lb/gal}\right)(3 \text{ gal/min})$$

$$= 6 \text{ lb/min} - \frac{3}{50} y \text{ lb/min.}$$

Since the tank initially contains no salt, $y(0) = 0$. The initial value problem for question (a) is thus

$$\frac{dy}{dt} = 6 - 0.06y, \qquad y(0)=0.$$

(b) Rewrite the differential equation as $\frac{dy}{dt} + 0.06y = 6$. This is a first order linear differential equation that we can solve with an integrating factor.

(Step 1) Find the integrating factor. $\mu(x) = e^{\int 0.06 dt} = e^{0.06t}$.

(Step 2) Multiply both sides of the differential equation by the integrating factor.

$$e^{0.06t}\left(\frac{dy}{dt} + 0.06y\right) = 6e^{0.06t}$$

$$e^{0.06t}\frac{dy}{dt} + e^{0.06t}(0.06)y = 6e^{0.06t}$$

(Step 3) Recognize that the left side is the derivative of a product of the integrating factor and *y*. (CHECK IT!)

$$\frac{d}{dt}\left[e^{0.06t}y\right] = 6e^{0.06t}$$

(Step 4) Integrate both sides with respect to *t* and then solve for *y*.

$$e^{0.06t}y = \int 6e^{0.06t}dt = \tfrac{6}{0.06}e^{0.06t} + C$$

$$e^{0.06t}y = 100e^{0.06t} + C$$

$$y = 100 + Ce^{-0.06t}$$

(Step 5) Apply the initial condition $y(0) = 0$ to determine the constant *C*.

$$0 = 100 + Ce^{-0.06(0)} = 100 + Ce^{0} = 100 + C \quad \Rightarrow \quad C = -100$$

The solution to the IVP is thus $y(t) = 100 - 100e^{-0.06t}$.