Calculus II – Math 2411-003
Exam 1 – Fall 2009

Name Key

Answer each question in the space provided. The exam is closed book and closed notes. **Graphics calculators are not allowed.** Partial credit is possible on most problems if you show your work. While the final answer is important, you earn points for the work leading to that answer, as well as the answer itself. Show all your steps clearly so you will be eligible for the most partial credit. You may not communicate with anyone other than the instructor during the exam. The instructor may help to clarify the meaning of a question, but will not give you hints as to the solution process. The exam is worth a total of 100 points. You will have until the end of the class period (1 hour and 50 minutes) to complete the exam. Good Luck!

1. (6 points each) Evaluate the integrals. Simplify the answers to definite integrals.

a) $\int_0^{\pi/6} 2\sin(3x)\,dx$

Let $u = 3x$

$\frac{du}{dx} = 3$

$\frac{du}{3} = dx$

$x = \pi/6 \Rightarrow u = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$

$x = 0 \Rightarrow u = 3(0) = 0$

$$= 2\int_0^{\pi/6} \sin(3x)\,dx$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin(u)\,du$$

$$= \frac{2}{3}\Big[-\cos(u)\Big]_0^{\pi/2}$$

$$= \frac{2}{3}\left[-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right]$$

$$= \frac{2}{3}[\,0 + 1\,]$$

$$= \frac{2}{3}$$

b) $\int_1^e \frac{\ln x}{x}\,dx$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

$x = e \Rightarrow u = \ln(e) = 1$

$x = 1 \Rightarrow u = \ln(1) = 0$

$$= \int_1^e \ln(x)\cdot\frac{1}{x}\cdot dx$$

$$= \int_0^1 u\,du$$

$$= \left[\frac{u^2}{2}\right]_0^1$$

$$= \left[\frac{1}{2} - 0\right]$$

$$= \frac{1}{2}$$

c) $\int \cos^4(x)\sin(x)\,dx \quad = \int (\cos x)^4 \sin x\,dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

$$= -\int u^4\,du$$

$$= -\frac{u^5}{5} + C$$

$$= -\frac{1}{5}\cos^5 x + C$$

d) $\int \frac{5+2x^4}{x}\,dx \quad = \int \left(\frac{5}{x} + \frac{2x^4}{x}\right) dx$

$$= \int \frac{5}{x}\,dx + 2\int x^3\,dx$$

$$= 5\ln|x| + 2\,\frac{x^4}{4} + C$$

$$= 5\ln|x| + \frac{x^4}{2} + C$$

e) $\int \frac{2}{\sqrt{1-9x^2}}\,dx \quad = \int \frac{2}{\sqrt{1-(3x)^2}}\,dx$

Let $u = 3x$

$\frac{du}{dx} = 3$

$\frac{du}{3} = dx$

$$= \int \frac{2}{\sqrt{1-u^2}}\,\frac{du}{3}$$

$$= \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{2}{3}\sin^{-1}(u) + C$$

$$= \frac{2}{3}\sin^{-1}(3x) + C$$

f) $\int x^2\sqrt{4+x^3}\,dx \quad = \int \sqrt{u}\,\frac{du}{3}$

Let $u = 4+x^3$

$\frac{du}{dx} = 3x^2$

$\frac{du}{3} = x^2 dx$

$= \frac{1}{3}\int u^{1/2}\,du$

$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right] + C$

$= \frac{2}{9}(4+x^3)^{3/2} + C$

g) $\int x^2 e^x\,dx$

$$\begin{array}{ccc} x^2 & & e^x \\ & \overset{(+)}{\searrow} & \\ 2x & & e^x \to x^2e^x \\ & \overset{(-)}{\searrow} & \\ 2 & & e^x \to -2xe^x \\ & \overset{+}{\searrow} & \\ 0 & & e^x \to +2e^x \end{array}$$

Tabular Integration

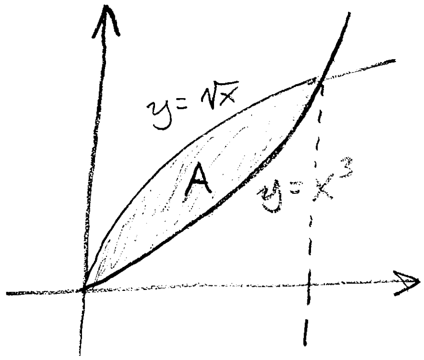
$\int x^2 e^x dx = x^2e^x - 2xe^x + 2e^x + C$

2. (7 points) Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

$g'(x) = \frac{d}{dx}\left[\int_0^x \sqrt{1+t^2}\,dt\right] = \sqrt{1+x^2}$ by the FTC.

3. Consider the region bounded by $y = \sqrt{x}$ and $y = x^3$.

a) (3 points) Sketch the region bounded by these curves.



$\sqrt{x} = x^3 \Rightarrow x = 0, 1$

b) (6 points) Find the area of the region bounded by these two curves.

$$A = \int_0^1 (\sqrt{x} - x^3)\,dx$$

$$= \int_0^1 (x^{1/2} - x^3)\,dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^4}{4}\right]_0^1$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^1$$

$$= \left(\frac{2}{3}(1)^{3/2} - \frac{1}{4}(1)^4\right) - (0)$$

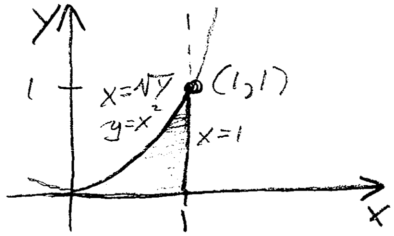
$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \frac{5}{12}$$

4. Consider the region bounded by $y = x^2$, $y = 0$, and $x = 1$.

a) (3 points) Sketch the region bounded by these curves.



$y = x^2$

$\sqrt{y} = x$

b) (5 points each) Set up, but **do not evaluate,** an integral you would use to find the volume of the solid of revolution formed by rotating the region about:

i. The y-axis using washers.

$$V = \pi \int_c^d \left((\text{outer radius})^2 - (\text{inner radius})^2 \right) dy$$

$$= \pi \int_0^1 \left[(1)^2 - (\sqrt{y})^2 \right] dy$$

$$= \pi \int_0^1 (1 - y)\, dy$$

ii. The y-axis using shells.

$$V = \int_c^d 2\pi (\text{radius})(\text{height})\, dx$$

$$V = \int_0^1 2\pi\, x\, (x^2)\, dx$$

$$= \int_0^1 2\pi\, x^3\, dx$$

iii. The line $y = -1$

By Washers:

$$V = \pi \int_a^b \left[(\text{outer radius})^2 - (\text{inner radius})^2 \right] dx$$

$$V = \pi \int_0^1 \left[(x^2 + 1)^2 - (1)^2 \right] dx$$

$$= \pi \int_0^1 (x^4 + 2x^2)\, dx$$

By Shells:

$$V = \int_a^b 2\pi (\text{radius})(\text{height})\, dy$$

$$V = \int_0^1 2\pi\, (y+1)(1 - \sqrt{y})\, dy$$

5. (6 PTS) Set up the integral that you would use for finding the arc length of the graph of $y = e^{x^2}$ from $x = 3$ to $x = 5$. **Do not evaluate** the integral.

$$L = \int_a^6 \sqrt{1 + [f'(x)]^2}\, dx \qquad f'(x) = \frac{d}{dx}[e^{x^2}] = 2xe^{x^2}$$

$$L = \int_3^5 \sqrt{1 + [2xe^{x^2}]^2}\, dx$$

or

$$= \int_3^5 \sqrt{1 + 4x^2 e^{2x^2}}\, dx$$

6. (6 PTS) Set up the integral that you would use to find the surface area generated by revolving the graph of the function $y = e^{3x}$ about the x-axis from $x = 1$ to $x = 2$. **Do not evaluate** the integral.

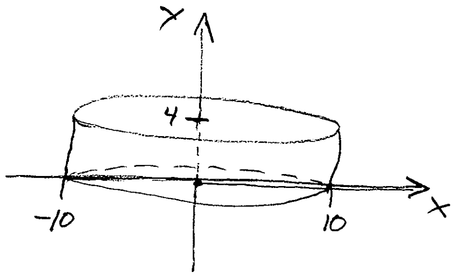
$$S = \int_a^6 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx \qquad f(x) = e^{3x}, \quad f'(x) = 3e^{3x}$$

$$= \int_1^2 2\pi e^{3x} \sqrt{1 + (3e^{3x})^2}\, dx$$

or

$$= 2\pi \int_1^2 e^{3x} \sqrt{1 + 9e^{6x}}\, dx$$

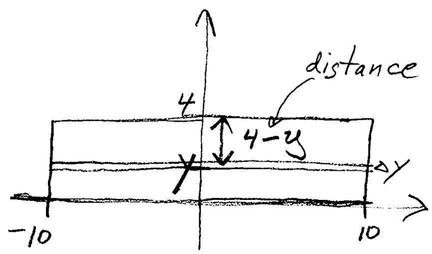
7. (6 PTS) A right circular cylindrical tank has a height of 4 feet and radius of 10 feet. If the tank is full of water, set up but **do not evaluate** the integral for finding the work done in pumping all of the water over the top of the tank. Water weighs 62.4 lbs/ft^3.



$$\text{Work} = \int_a^6 (\text{volume of thin disks})(62.4)(\text{distance})\,dv$$

Volume of Disk is $\pi r^2 \Delta y = \pi(10)^2 \Delta y = 100\pi \Delta y$

Distance is $4 - y$



$$\text{Work} = \int_0^4 100\pi(62.4)(4-y)\,dy$$

8. (6 PTS) A force of 5 pounds is required to stretch a spring 4 inches from its natural length. Find the work done in stretching the spring 6 inches from its natural length. Include units in your answer.

$F = kx$

$5 = k4$

$\frac{5}{4} = k$

$F(x) = \frac{5}{4}x$

$$W = \int_0^6 \frac{5}{4}x\,dx$$

$$= \frac{5}{4} \cdot \frac{x^2}{2}\Big]_0^6$$

$$= \frac{5}{4}\left[\frac{36}{2} - 0\right]$$

$$= \frac{45}{2} \text{ in-pounds}$$

$F = kx$

$5 = k\left(\frac{1}{3}\right)$

$15 = k$

$F(x) = 15x$

$$W = \int_0^{1/2} 15x\,dx$$

$$= \frac{15}{2}x^2\Big]_0^{1/2}$$

$$= \frac{15}{2}\left[\frac{1}{4} - 0\right]$$

$$= \frac{15}{8} \text{ ft-lbs}$$