Math 5663 (Introduction to Finite Element Methods) **Fall 1998**

NAME: ______________________________

TAKE-HOME FINAL

Help from anybody constitutes cheating. Open books, notes, homeworks. You may use calculators or notebooks. Guessed answers are NOT accepted. Good luck!

1) (The following equation numbers refer to equations in Johnson's book) For the abstract variational problem given in eq. (2.5) satisfying conditions (i) through (iv), approximated as in eq. (2.9), prove the following sharper error estimate

$$||u - u_h||_V \leq \sqrt{\frac{\gamma}{\alpha}}\, ||u - v||_V \qquad \forall v \in V_h \,.$$

Why is it sharper?

2) Consider the following boundary value problem defined in the unit interval, $\Omega = (0, 1)$:

$$(D) \quad \begin{cases} -u'' - k^2 u = 0 \quad \text{in } \Omega \\ u(0) = 0 \\ u'(1) = 1 \end{cases}$$

where k is a given constant. The following questions refer to this problem.

2.1) Set up a variational formulation (V) for (D).

2.2) Comparing with the following abstract variational formulation:

$$(V) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) = L(v) \qquad \forall v \in V, \end{cases}$$

give the explicit formulas for V, $a(u, v)$ and $L(v)$ for the variational formulation found in item 2.1).

2.3) From now on let us fix $k = 1/2$. Do the four conditions (i) through (iv) given in page 50 hold for this problem? If yes, find the constants γ, α and Λ.

2.4) Let us now partition the unit interval uniformly into N elements (i.e., $h = 1/N$) and consider the usual continuous piecewise linear functions as the finite element subspace $V_h \subset V$.

2.4.1) Compute the element stiffness matrix coefficients and element loads for any element of our uniform partition.

2.4.2) Compute the global stiffness matrix $\mathbf{A}$ for $N = 2, 4, 8$ and 16. (As usual, consider the node numbering done sequentially along the interval).

2.4.3) Compute the solution u_h of the finite element method for $N = 2, 4, 8$ and 16.

2.4.4) Compute the analytical solution u of (D).

2.4.5) Use the results of the previous two itens to compute the H_1-norm of the error for $N = 2, 4, 8$ and 16. Plot your results on a log-log graph for the H_1-norm of the error versus the number of elements. Does the error behave as expected as the number of elements increase? Explain.