Math 7172 (Introduction to Finite Element Methods) **Fall 1996**

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TAKE-HOME FINAL: DUE December 12, 2 pm in room 620 E.

Help from anybody constitutes cheating. Open books, notes, homeworks. You may use calculators or notebooks. Guessed answers are NOT accepted. Good luck!

Let us consider the following boundary value problem defined on the unit interval, $\Omega = (0,1)$:

$$(D) \quad \begin{cases} -\epsilon^2 u_{,xx} + \sigma^2 u = 0 & \text{in } \Omega \\ u(0) = 0; u(1) = 1 \end{cases}$$

where ϵ and σ are given constants, and the unknown u is a scalar valued function on Ω, i.e., $u : \Omega \to \mathbb{R}$. Comma subscript denotes differentiation with respect to x.

1) Set up a variational formulation for (D) with functions that are square-integrable on Ω and have derivative that is also square-integrable on Ω.

2) Consider now the standard Galerkin method for the variational formulation obtained in item 1), defined on the set of functions that are piecewise linear and continuous with appropriate conditions on the endpoints of the domain. These set of functions are defined on a uniform partition T_h of Ω, i.e., each element diameter is $h_K = h = 1/N$ where N is the number of elements (or subintervals) of our partition, and has as parameters the values at the node points of T_h.

2.1) Based on the matrix formulation equivalent to the Galerkin method ($\mathbf{A}\xi = \mathbf{b}$), compute the element stiffness matrix coefficients and element loads for any element of our uniform partition.

2.2) Sketch the entries of the global stiffness matrix $\mathbf{A}$ (i.e., assemble the element stiffness matrices into the global $\mathbf{A}$) if we consider the node numbering done sequentially along the interval.

2.3) For a typical interior node I, give the finite element formulas associated with the unknown ξ_I. (Hint: multiply row I of $\mathbf{A}$ with the unknown vector ξ and equate to the I-th component of $\mathbf{b}$. The resulting equation should read $a\xi_{I-1} + b\xi_I + c\xi_{I+1} = d$ where a, b, c, d are coefficients for you to determine in terms of the given data and h.)

2.4) Divide the finite difference equation obtained in item 2.3) for a typical interior node I by a coefficient so that the coefficients a, b and c of the finite difference-like equation obtained in item 2.3), i.e.,

$$a\xi_{I-1} + b\xi_I + c\xi_{I+1} = d$$

has coefficient $a = (1 - \alpha)$ where

$$\alpha = \frac{\sigma^2 h^2}{6\epsilon^2}\,. \tag{1}$$

2.5) The general solution for the finite difference equation of item 2.4) can be expressed as

$$\xi_I = cr^I.$$

Substitute this expression into the finite difference equation obtained in 2.4) (clearly, $\xi_{I-1} = cr^{I-1}$ and $\xi_{I+1} = cr^{I+1}$) and divide by cr^{I-1}. You should obtain a second order algebraic equation for r. Find the two roots r_1 and r_2.

2.6) Examine the exact solution of our problem (D) when $\epsilon \ll 1$. Sketch the plot of the exact solution in this case ($u = u(x)$). The roots found in item 2.5) leads to the discrete equation for the coefficients,

$$\xi_I = C_1 r_1^I + C_2 r_2^I\,.$$

How well will the Galerkin solution ($u_h(x) = \sum_I \varphi_I(x)\xi_I$, where φ_I are the usual basis functions for the piecewise linear finite element space) approximate the exact solution sketched? You are supposed to give a qualitative answer (not a quantitative one!) and justify it.

3) Consider now stabilization of the previous formulation, by adding to the Galerkin method the mesh-dependent term:

$$\sum_K \int_K (\sigma^2 u_h - \epsilon^2 u_{h,xx})_{,x} \tau (\sigma^2 v - \epsilon^2 v_{,xx})_{,x}\, dx$$

where v is the test function, u_h the trial function, both defined on sets of function that are continuous piecewise linear and with appropriate boundary conditions. We consider uniform mesh herein. Note that this additional term is the derivative least-squares form of the original equation. The parameter τ is defined to be

$$\tau = \frac{h^2}{6\sigma^2}\tilde{\xi}\,. \tag{2}$$

3.1) Find $\tilde{\xi}$ as a function of α such that we obtain a nodally exact solution with this stabilized formulation for (D). (*Hint:* Look at the derivation of $\tilde{\xi}$ for the exact artificial diffusion method for advective diffusive equations. The procedure should be similar to that derivation).

3.2) Instead of using the exact formula for $\tilde{\xi}$ derived in the previous item, we will use the asymptotic expression for $\tilde{\xi}$ given by

$$\tilde{\xi} = \begin{cases} 1, & \alpha \geq 8, \\ 0.064\alpha + 0.49, & 1 \leq \alpha < 8, \\ 0, & \alpha < 1. \end{cases} \tag{3}$$

Let $\tilde{u}_h$ denote the interpolant of the exact solution. Using equations (1)-(3), show that

(i) if $\alpha \geq 1$ then

$$|||\tilde{u}_h - u|||^2 \leq \sigma^2 C(u) h^{2k+2}$$

(ii) if $\alpha < 1$ then

$$|||\tilde{u}_h - u|||^2 \leq \epsilon^2 C(u) h^{2k}$$

where $C(u)$ is a constant only depending on the exact solution u, assuming different values in different appearances and the $|||\cdot|||$ norm is defined as

$$|||v||| \equiv \left(\sigma^2||v||_0^2 + \epsilon^2||v_{,x}||_0^2 + \sum_K \tau||\sigma^2 v_{,x} - \epsilon^2 v_{,xxx}||_{0,K}^2\right)^{1/2} \tag{4}$$

3.3) Prove stability of the bilinear form for this stabilized method with respect to the $|||\cdot|||$ norm given in equation (4).

3.4) Prove that the error in this stabilized method converge as follows:

(i) if $\alpha \geq 1$ then

$$|||u_h - u|||^2 \leq \sigma^2 C(u) h^{2k+2}$$

(ii) if $\alpha < 1$ then

$$|||u_h - u|||^2 \leq \epsilon^2 C(u) h^{2k}$$

3.5) Based on the error estimate derived in the previous item, can we assert that the finite element solution converges with optimal rate in the H^1 seminorm, independent of α? Justify your answer.