Math 5661 (Numerical Analysis II) **Spring 1995**

NAME: ______________________________

MIDTERM

Open books, notes, homeworks. You may use calculators or notebooks. Guessed answers are NOT accepted. Good luck!

1) Consider the problem of evaluating the integrals of the type

$$I = \int_{-1}^{1} f(x)(1-x^2)^{-1/2}\, d\, x$$

The family of orthogonal polynomials associated with $w(x) = (1-x^2)^{-1/2}$ are the Tchebycheff polynomials given by $\varphi_0 = 1, \varphi_1 = x, \varphi_2 = 2x^2 - 1, \varphi_3 = 4x^3 - 3x, \ldots$

(7 pt) a) Derive a Gauss quadrature rule that is exact for all linear polynomials. *Hint:* $\int_{-1}^{1}(1-x^2)^{-1/2}\, d\, x = \pi$

(3 pt) b) Using the quadrature rule derived in the previous item, compute

$$I = \int_{-1}^{1} (5x-2)(1-x^2)^{-1/2}\, d\, x$$

2) Consider the following initial value problem

$$\begin{aligned} y' &= -20y \qquad 0 < x < 1 \\ y(0) &= 1 \end{aligned}$$

(5 pt) a) What is the bound on the step size such that the Forward Euler is stable?

(5 pt) b) If we take $h = 0.1$, will the fourth-order Runge-Kutta method be stable for this problem? Justify.

(5 pt) c) For the Backward Euler method, what is the range of step sizes such that we have stability?

3) Consider a uniform partition of the interval $[a, b]$ into N subintervals of size $h = (b-a)/N$, $x_i = a + ih$.

(5 pt) a) Consider now functions with continuous values and continuous first three derivatives, to show that we can compute the first derivative at x_i using:

$$y'(x_i) = \frac{-3y(x_i) + 4y(x_{i+1}) - y(x_{i+2})}{2h} + ch^p y'''(\xi_i)$$

for some ξ_i in (x_i, x_{i+2}). Find the values of p and c in the formula above *(Hint: Use Taylor's formula to expand $y(x_{i+1})$ and $y(x_{i+2})$ around $y(x_i)$)*

(10 pt) b) Part a) suggests the following difference method

$$y_{n+2} = 4y_{n+1} - 3y_n - 2hf(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

to approximate the well-posed problem

$$\begin{aligned} y' &= f(x, y) \qquad a < x < b \\ y(a) &= \alpha \end{aligned}$$

b.1) Classify the difference method: (Mark what applies)

I) one-step method

II) multi-step method

III) explicit method

IV) implicit method

b.2) For the model problem with $f(x, y) = 0$, classify this difference method according to its stability characteristics.