Math 5661 (Numerical Analysis II) **Spring 1998**

NAME: ______________________________

FINAL

Help from anybody constitutes cheating. Open books, notes, homeworks. You may use calculators or notebooks. Guessed answers are NOT accepted. Good luck!

1) *Central and Upwind Finite-Difference Methods for Advective-Diffusive Equations*

Advective-diffusive equations model transport phenomena such as the transport of mass, momentum, or energy in a flow field, the transport of different species or chemical substances, etc. A simple model that describes these phenomena is given by

$$u(x)y'(x) - \kappa y''(x) = 0 \qquad 0 < x < L \tag{1}$$

$$y(0) = 1 \qquad y(L) = 0 \tag{2}$$

where $u(x)$ is a given velocity field, κ is a given positive constant called diffusivity and $y(x)$ is the quantity being transported to be determined in the domain $0 < x < L$.

Let us further simplify this model and consider a given positive constant velocity field, i.e.,

$$u(x) = u > 0 \qquad 0 < x < L \tag{3}$$

(10 pt) a) Find the analytical solution of this problem employing a technique from your elementary ODE course. Express your answer in terms of the non-dimensional number Peclet given by

$$Pe = \frac{uL}{\kappa} \tag{4}$$

Next consider a partition of $[0, L]$ into N uniform intervals of length h, i.e., $h = L/N$, and for $k = 0, 1, \ldots, N$ set

$$x_k = kh$$

If we denote by w_k (or w_k^U used for the upwind method) the approximation (to be computed) for $y(x_k), k = 1, 2, \ldots, N-1$ and set $w_0 = y(0) = 1, w_N = y(L) = 0$, then we consider approximating $y''(x_k)$ by

$$y''(x_k) \approx \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} \tag{5}$$

and $y'(x_k)$ by two different ways, namely, centered-formula

$$y'_C(x_k) \approx \frac{w_{k+1} - w_{k-1}}{2h} \tag{6}$$

and upwind formula

$$y'_U(x_k) \approx \frac{w_k - w_{k-1}}{h} \tag{7}$$

Using the centered formulas (5) and (6) in (1) yields the central differences method

$$\frac{u}{2h}(w_{k+1} - w_{k-1}) = \frac{\kappa}{h^2}(w_{k+1} - 2w_k + w_{k-1}) \qquad \text{for } k = 1, 2, \ldots, N-1 \tag{8}$$

Introducing the mesh Peclet number

$$\alpha = \frac{uh}{2\kappa} \tag{9}$$

eq. (8) can be rewritten as

$$(1-\alpha)w_{k+1} - 2w_k + (1+\alpha)w_{k-1} = 0 \qquad \text{for } k = 1, 2, \ldots, N-1 \tag{10}$$

Using the formulas (5) and (7) in (1) yields the upwind method

$$\frac{u}{h}(w_k^U - w_{k-1}^U) = \frac{\kappa}{h^2}(w_{k+1}^U - 2w_k^U + w_{k-1}^U) \qquad \text{for } k = 1, 2, \ldots, N-1 \tag{11}$$

Or, by using (9), eq. (11) can be recast in the form

$$w_{k+1}^U - 2(1+\alpha)w_k^U + (1+2\alpha)w_{k-1} = 0 \qquad \text{for } k = 1, 2, \ldots, N-1 \tag{12}$$

(20 pt) b) Find the general solution to (10) and to (12).

(15 pt) c) For $\alpha > 1$ show that one of the roots of the auxiliary equation for the method (10) becomes negative, whereas for method (12) both roots are always positive for any $\alpha > 0$.

(15 pt) d) For $u = 1, \kappa = 10^{-6}, L = 1$, and $h = 0.2$, use methods (10) and (12) to approximate the solution to (1)-(2). Compare these approximations with the actual solution obtained in part (a). Note the spurious oscillations for method (10) from one point to the next. Repeat the exercise with $h = 0.04$.

(This exercise illustrates that numerical oscillations occur for certain methods, and alternative methods should be considered for an appropriate simulation of the actual problem)

2) *Least-squares approximation of functions using FEM machinery*

Consider the least-squares problem of approximating a given function $f(x) \in L_2(I)$ defined on the unit interval $I = (0,1)$, by a continuous piecewise-linear polynomial function u_h. This problem fits the framework of the finite element method we discussed. The continuous problem can be written as:

$$(V) \quad \begin{cases} \text{Find } u \in L_2(I) \text{ such that} & \\ (u,v) = (f,v) & \forall v \in L_2(I) \end{cases}$$

where as usual $(\,\cdot\,,\,\cdot\,)$ denotes the $L_2(I)$ inner product (i.e., integral over I).

The following questions refer to this problem.

(5 pt) 2.1) If $f \in L_2(I)$, can you find the solution u to (V)? Justify your answer.

(5 pt) 2.2) Comparing with the following abstract variational formulation:

$$(V) \quad \begin{cases} \text{Find } u \in V \text{ such that} & \\ a(u,v) = L(v) & \forall v \in V, \end{cases}$$

give the explicit formulas for V, $a(u,v)$ and $L(v)$ for the variational formulation.

(10 pt) 2.3) Do the four conditions (i) through (iv) given in page 50 of Johnson hold for this problem? If yes, find the constants γ, α and Λ and give an error estimate for this problem.

2.4) Let us now partition the unit interval uniformly into N elements (i.e., $h = 1/N$) and consider the usual continuous piecewise linear functions as the finite element subspace $V_h \subset V$. From now on consider $f = \sin(\pi x)$.

(5 pt) 2.4.1) Does the best approximation result hold for this problem? Explain your answer.

(5 pt) 2.4.2) Compute the element stiffness matrix coefficients.

(5 pt) 2.4.3) Compute the global stiffness matrix $\mathbf{A}$ for $N = 2, 4$ and 8 elements. (As usual, consider the node numbering done sequentially along the interval).

(10 pt) 2.4.4) Compute the global load vectors $\mathbf{b}$ for $N = 2, 4$ and 8 elements. Use two-point Gaussian quadrature rules for Legendre polynomials to approximate the integration, or compute them analytically.

(5 pt) 2.4.5) Compute the solution u_h of the finite element method for $N = 2, 4$ and 8.

(10 pt) 2.4.6) Compute the L_2-norm of the (true) error for $N = 2, 4$ and 8. Explain the error behavior as the number of elements increase.