Math 5661 (Numerical Analysis II) **Spring 1997**

NAME: ______________________________

FINAL

Help from anybody constitutes cheating. Open books, notes, homeworks. You may use calculators or notebooks. Guessed answers are NOT accepted. Good luck!

(20 pt) 1) Find the general (analytical) solution for the IVP:

$$
\begin{aligned}
x'' + \frac{(x')^2}{x} &= 0 \\
x(0) &= 1 \\
x'(0) &= z
\end{aligned}
$$

in terms of z and of the independent variable t. Use this solution along with the shooting method procedure to obtain the analytical solution for the BVP:

$$
\begin{aligned}
x'' + \frac{(x')^2}{x} &= 0 \\
x(0) &= 1 \\
x(1) &= 2
\end{aligned}
$$

(25 pt) 2) In the computer problem 9.1 assigned for homework # 5 you are first asked to use $h = 0.1$ and $k = 0.005125$, which implies that $s > 0.5$. The numerical result you obtained for these choices of parameters produced a stable numerical result. Explain the apparent contradiction between this result and the statement at the end of that section that $s \leq 0.5$ for stability.

(30 pt) 3) To approximate the parabolic problem given in eq. (2), pg 664, KC, we consider the following methods (using the notation in KC):

a) $$\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} = \frac{v_{i,j+1} - v_{i,j-1}}{2k}$$

b) $$\frac{v_{i+1,j} - (v_{i,j+1} + v_{i,j-1}) + v_{i-1,j}}{h^2} = \frac{v_{i,j+1} - v_{i,j-1}}{2k}$$

Classify and justify each method as either unconditionally stable, or conditionally stable, or unconditionally unstable (*Hint: Use the Fourier method, page 670*).

4) Consider the following boundary value problem defined in the unit interval, $\Omega = (0,1)$:

$$(D) \quad \begin{cases} -u'' = 0 & \text{in } \Omega \\ u(0) = 0 \\ u'(1) = g \end{cases}$$

where g is a given constant. The following questions refer to this problem.

(5 pt) 4.1) Set up a variational formulation (V) for (D).

(5 pt) 4.2) Comparing with the following abstract variational formulation:

$$(V) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = L(v) \qquad\qquad \forall v \in V, \end{cases}$$

give the explicit formulas for V, $a(u,v)$ and $L(v)$ for the variational formulation found in item 4.1).

(10 pt) 4.3) Do the four conditions (i) through (iv) given in page 50 of Johnson hold for this problem? If yes, find the constants γ, α and Λ and give an error estimate for this problem.

4.4) Let us now partition the unit interval uniformly into N elements (i.e., $h = 1/N$) and consider the usual continuous piecewise linear functions as the finite element subspace $V_h \subset V$.

(5 pt) 4.4.1) Compute the element stiffness matrix coefficients and element loads for any element of our uniform partition.

(5 pt) 4.4.2) Compute the global stiffness matrix $\mathbf{A}$ for $N = 2$ and 4 elements. (As usual, consider the node numbering done sequentially along the interval).

(5 pt) 4.4.3) Compute the solution u_h of the finite element method for $N = 2$ and 4.

(5 pt) 4.4.4) Compute the H_1-norm of the (true) error for $N = 2$ and 4. Explain the error behavior as the number of elements increase.

(5 pt) 4.4.5) How does the error behave if we replace g by $2g$? Explain the error behavior as the number of elements increase in this case.