

NAME: \_\_\_\_\_

**FINAL**

2 hours long. Open books, notes, homeworks. Calculators and/or notebooks, pcs are NOT allowed. You may leave an answer in the form  $2^{10}/9!$ , for example. Straightforward computations should be done! Gessed answers are NOT accepted. Good luck!

(20 pt) 1) Write down the quadratic polynomial which interpolates  $\sqrt{x}$  at the points  $x = 1, 4$  and  $9$  using Newton's form of the interpolating polynomial.

(20 pt) 2) Is the following function a natural cubic spline? Justify your answer.

$$S(x) = \begin{cases} x^3 - 1, & x \in [-1, 0] \\ 3x^3 - 1, & x \in [0, 1] \end{cases}$$

(20 pt) 3) Compute two steps of the power method with  $\varphi(x) = x_1$  (i.e., the first component of  $x$ ) on the following matrix and initial vector:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Is it converging as expected? Explain your answer. If your previous answer is negative, then suggest some way to fix the usage of the algorithm.

(20 pt) 4) Find the least squares straight line for  $f(x) = \sqrt{1-x^2}$  on  $[-1, 1]$  using the inner product

$$\langle f, g \rangle = \int_{-1}^1 \frac{fg}{\sqrt{1-x^2}} dx.$$

*Hint: See example 3 on page 430 of the textbook.*

(20 pt) 5) Let  $A$  be a nonsingular  $n \times n$  matrix,  $b$  an  $n$ -vector,  $x$  the exact solution to  $Ax = b$ , and  $x^*$  an approximate solution. Suppose that  $\|A\| = 6$ ,  $\|A^{-1}\| = 3$ ,  $\|b\| = 10$ ,  $\|x^*\| = 3$  and  $\|Ax^* - b\| = 0.001$  (where the  $\infty$ -norm is used for all norms). What is the largest possible value of the relative error  $\|x - x^*\|/\|x\|$ ?

(20 pt) 6) Find conditions on the number  $\alpha$  which assure that the iteration  $x_{n+1} = x_n - \alpha f(x_n)$  will converge to a zero of  $f$  (say  $f(s)$ ) for any starting value sufficiently near the root. Sketch geometrically two iterations with this method.