Adaptive-Multilevel BDDC

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ABSTRACT

The Balancing Domain Decomposition by Constraints (BDDC) method proposed by Clark R. Dohrmann [1] is the most advanced method from the balancing family of iterative substructuring methods for the solution of large systems of linear algebraic equations arising from discretization of elliptic boundary value problems. In the case of many substructures, solving the coarse problem exactly becomes a bottleneck. Since the coarse problem in BDDC has the same structure as the original problem, it is straightforward to apply the BDDC method recursively to solve the coarse problem only approximately. We have formulated a new family of abstract Multispace BDDC methods and gave condition number bounds from the abstract additive Schwarz preconditioning theory. The Multilevel BDDC is then treated as a special case of the Multispace BDDC. However, the condition number bounds reveal deteriorating convergence of the method with increasing number of levels that cannot be improved. In another research direction we have developed a method for the adaptive selection of the coarse space for the original, two-level, BDDC method. The method works by adding coarse degrees of freedom constructed from eigenvectors associated with intersections of selected pairs of adjacent substructures. In this contribution we combine the advantages of both approaches to propose a method that preserves both, parallel scalability with increasing number of subdomains and excellent convergence properties. Performance of the method is illustrated by numerical examples.

1 The BDDC method

The Balancing Domain Decomposition by Constraints (BDDC) can be viewed as a preconditioner for the Conjugate Gradient (CG) method used for iterative solution of systems of linear equations.
Let the domain Ω be decomposed into N nonoverlapping subdomains Ω^i, $i = 1, ..., N$, and let us consider an abstract variational problem

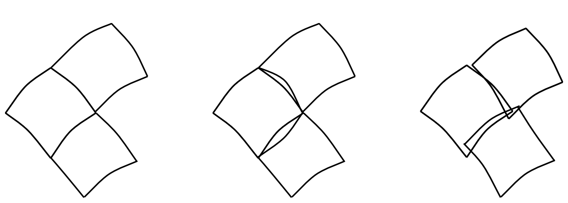
$$a(u, v) = \langle f, v\rangle \quad \forall v \in \widehat{W}, \tag{1}$$

where $\widehat{W}$ is a finite dimensional space, $u \in \widehat{W}$ is the solution to be found, and $f \in \widehat{W}'$ is the right hand side. By $\widehat{W}'$, we denote the dual space to $\widehat{W}$. An equivalent formulation of (1) is to find a solution u to a linear system

$$Au = f, \tag{2}$$

where A is the stiffness matrix, reduced to substructure interfaces. Let W_i be the space of finite element functions on subdomain Ω_i and define $W = W_1 \times \cdots \times W_N$. The BDDC is characterized by selection of *coarse degrees of freedom*, such as values at the corners and averages over edges or faces of subdomains. In the present setting, this becomes the selection of the subspace $\widetilde{W} \subset W$, defined as the subspace of all functions such that coarse degrees of freedom are continuous across the interfaces. There needs to be enough coarse degrees of freedom that the variational problem on $\widetilde{W}$ is coercive. Figure 1 presents schematic drawing of continuity conditions between substructures, in the case of corner coarse degrees of freedom only: all degrees of freedom continuous (the space $\widehat{W}$), only the coarse degrees of freedom need to be continuous (the space $\widetilde{W}$), and no continuity conditions (the space W).

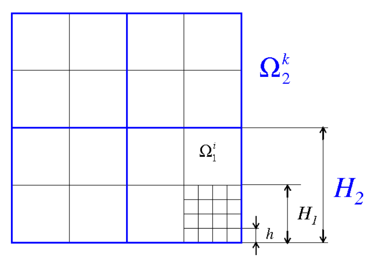
Figure 1



In each iteration, the action of the BDDC preconditioner consists of: (i) distributing the residual into the larger space $\widetilde{W}$, (ii) solving the problem in the space $\widetilde{W}$, (iii) projecting the solution in the space $\widehat{W}$.

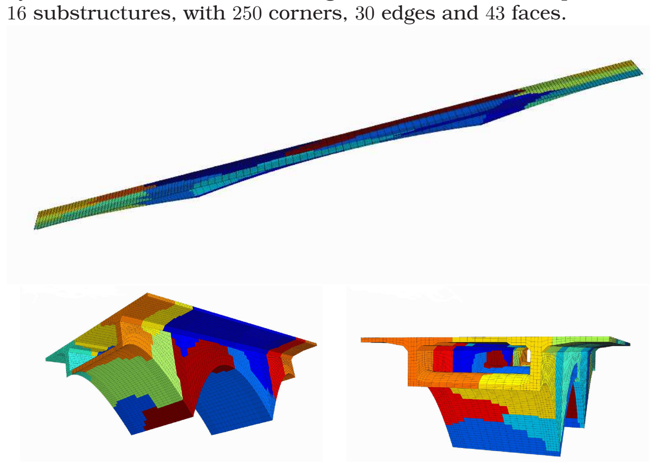
2 Multilevel BDDC

The idea of Multilevel BDDC [5, 6] is to apply the two-level method recursively, i.e., regard the substructures as elements and apply BDDC on each decomposition level as on the level one. An example of a decomposition for three-level method follows:



3 Adaptive algorithm

The method is based on the abstract condition number bound introduced by Mandel, Dohrmann and Tezaur in [2]. The adaptive algorithm was proposed in [3, 4] for two-dimensions, and reformulated and extended into three dimensions in [7]. We illustrate the performance of the method on a bridge construction discretized by finite elements with 157356 degrees of freedom, decomposed into 16 substructures, with 250 corners, 30 edges and 43 faces.



Results of the original method:

constraint	Nc	κ	it
c	750	2301.37	241
c+e	840	2252.41	237
c+e+f	969	653.61	167
c+e+f (3eigv)	969	177.77	108

The first two rows correspond to nonadaptive approach with corner constraints and arithmetic averages over edges/faces, and the last row corresponds to corner constraints with arithmetic averages over edges and three adaptive averages over faces.

Results of the adaptive method:

τ	$\widetilde{\omega}$	Nc	κ	it
∞(=c+e)	-	840	2252.41	237
650	589.338	845	483.52	178
30	29.568	952	28.74	64
5	4.997	1315	5.01	26
2	1.998	1961	2.01	14

Here τ is the desired condition number, $\widetilde{\omega}$ is the condition number indicator from the adaptive algorithm, Nc is the number of additional constraints, κ is the condition number estimate from CG, and finally it is the number of iterations needed to achieve $tol = 10^{-8}$.

4 Adaptive-Multilevel BDDC

The drawback of the Multilevel BDDC is that the condition number grows proportionally to the number of levels, and theory indicates that using the standard approach the condition number bound cannot be improved. For this reasons, we have recently implemented the adaptive selection of constraints in the Multilevel BDDC.
The algorithm was tested on a planar elasticity problem discretized by standard bilinear finite elements with 1182722 degrees of freedom, 2304 subdomains, each one consisting of 256 elements (H/h=16). We have compared two and three-level methods with one edge jagged on each decomposition level and corners as initial constraints. The results are summarized in the following table:

levels	τ	$\widetilde{\omega}$	Nc (level 1+2)	κ	it
2	∞(=c)	-	4794	18.41	43
2	2	1.972	18305	1.97	13
3	∞(=c)	-	4794+24	67.49	74
3	2	1.972	18305+117	2.28	15

5 Conclusions & Outlook

The presented numerical results are very promising. In the near future, we are planning to test the Adaptive-Multilevel BDDC on practical engineering problems with millions of degrees of freedom. Besides developing and testing new algorithms, another interest is their practical implementation on parallel computers using Message Passing Interface (MPI). This is an independent ongoing research conducted with a research group of Professor Pavel Burda from the Czech Technical University in Prague, e.g., [8].

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