Exchangeable Sequences, Laws of Large Numbers, and the Mortgage Crisis.

Myung Joo Song
Advisor: Prof. Jan Mandel

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Introduction

The law of large numbers for i.i.d. sequence gives convergence of sample means to a constant, i.e., a deterministic quantity.

The yield from a mortgage can be understood as a random variable. If a bank can make a large number of mortgages such their yields are i.i.d., the average yield will converge to a deterministic quantity.

But events that appear to be independent may in fact be only exchangeable.

In a real financial market, there is always a fundamental factor which can affect all the events at the same time, such as a war, or an economic crisis which ruins the assumption of independence.

Independence of random variables

Definition:

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Then, the function $X : \Omega \to \Re$ is a *(real - valued) random variable* if

$$\{\omega : X(\omega) \leq r\} \in \mathscr{F} \qquad \forall r \in \Re.$$

Two random variables X and Y are *independent* iff

$$\mathrm{P}(X \leq a, Y \leq b) = \mathrm{P}(X \leq a) \cdot \mathrm{P}(Y \leq b) \quad \forall a, b$$

then also, $\mathrm{E}\{(XY)\} = \mathrm{E}\{X\} \cdot \mathrm{E}\{Y\}$

note: independent $\Rightarrow$ uncorrelated ($\mathrm{cov}(X,Y) = 0$)

but the converse is not true.

Laws of Large Numbers

Weak Law of Large Numbers:

Given $\left(X_1, X_2, \cdots\right)$ an infinite sequence of $i.i.d.$ r.v.s with

$E\left(X_i\right) = \mu < \infty,\ \forall i \in \mathrm{N},$

$$\overline{X}_n = \frac{1}{n}\left(X_1 + \cdots + X_n\right) \overset{P}{\rightarrow} \mu \qquad \text{as } n \rightarrow \infty$$

That is, $\lim\limits_{n \to \infty} P\left(\left|\overline{X}_n - \mu\right| < \varepsilon\right) = 1$ for any ε.

Laws of Large Numbers

Strong Law of Large Numbers:

Given $\left(X_1, X_2, \cdots\right)$ an infinite sequence of $i.i.d.$ r.v.s with $E\left(X_i\right) = \mu < \infty,\ \forall i \in \mathrm{N},$

$$\bar{X}_n = \frac{1}{n}\left(X_1 + \cdots + X_n\right) \overset{a.s.}{\rightarrow} \mu \qquad \text{as } n \rightarrow \infty$$

That is, $P\left(\lim_{n\rightarrow\infty} \bar{X}_n = \mu\right) = 1.$

Exchangeability

Definition:

An infinite sequence of $X_1, \cdots, X_n, \cdots$ of random variables is said to be *exchangeable* if $\forall n \geq 2$,

$$\left(X_1, \cdots, X_n\right) \overset{D}{=} \left(X_{\pi(1)}, \cdots, X_{\pi(n)}\right) \quad \forall \pi \in S(n),$$

where $S(n)$ is the group of permutations of $\left\{1, \cdots, n\right\}$.

Exchangeability

Example: Polya's urn

An urn has initially r red and b black balls. Draw a ball at random and note its color, and replace the ball back and add another ball of the same color.

Let X_i=1 if the i^{th} draw yields a red ball and X_i=0 otherwise.

Then,

$$P(1,1,0,1)=\frac{r}{b+r}\cdot\frac{r+1}{b+r+1}\cdot\frac{b}{b+r+2}\cdot\frac{r+2}{b+r+3}$$

$$=\frac{r}{b+r}\cdot\frac{b}{b+r+1}\cdot\frac{r+1}{b+r+2}\cdot\frac{r+2}{b+r+3}=P(1,0,1,1).$$

Similarly for other cases.

The sequence $X_1,\cdots,X_n,\cdots$ is exchangeable.

Conditional Expectation

Let X be r.v. on probability space (Ω, S, P). Given another σ-algebra $\mathscr{F} \subset S$, a r.v. Y is called conditional expectation of X given $\mathscr{F}$ if Y is $\mathscr{F}$-measurable, that is,

$$\{\omega \in \Omega : Y(\omega) \le a\} \in \mathscr{F} \qquad \forall a \in \Re$$

and

$$\int_A Y dP = \int_A X dP \qquad \forall A \in \mathscr{F} \ .$$

In this case, denote $Y = \mathrm{E}(X \mid \mathscr{F})$.

Roughly speaking, $\mathrm{E}(X \mid \mathscr{F})$ is averaging of X to the granularity of $\mathscr{F}$ (if $\mathscr{F}$ is finite, averaging on the atoms of $\mathscr{F}$).

LLN for exchangeable sequences

Let $\left(\mathscr{F}_n\right)_{n\geq 0}$ be a sequence of σ-algebras on $\left(\Omega, \mathscr{F}, P\right)$. $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ $\forall n \geq 0$. Then, a sequence of r.v.s $\left(X_n\right)_{n\geq 0}$ is called a *martingale* if,

(i) $\mathrm{E}\left\{\left|X_n\right|\right\} < \infty$, each n;

(ii) X_n is $\mathscr{F}_n$-measurable, each n;

(iii) $\mathrm{E}\left\{X_n \mid \mathscr{F}_m\right\} = X_m$ *a.s.*, each $m \leq n$.

Martingale Convergence Theorem:

Let $\left(X_n\right)_{n\geq 1}$ be a martingale s.t. $\sup_n \mathrm{E}\left\{\left|X_n\right|\right\} < \infty$.

Then $\lim_{n\to\infty} X_n = X$ exists *a.s.* (*and is finite a.s.*). Moreover, X is in L^1.

LLN for exchangeable sequences

Recall

$\{X_i\}$ i.i.d., $\mathrm{E}\left[|X_1|\right] < +\infty$

$$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{N} X_i \;\to\; \mathrm{E}\left[X_1\right] \text{ a.s. , a deterministic number.}$$

Theorem:

$\{X_i\}$ exchangeable, $\mathrm{E}\left[|X_1|\right] < +\infty$

$$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{N} X_i \;\to\; \mathrm{E}\left[X_1 \mid \mathscr{F}\right] \text{ a.s. , a random variable for some } \mathscr{F}$$

LLN for exchangeable sequences

proof : Let an infinite sequence $X = (X_1, X_2, \cdots)$ of random variables be exchangeable and Let $\mathscr{F}_n$ be the σ-algebra generated by all the *n*-symmetric functions of X. $\mathscr{F}_n \supseteq \mathscr{F}_{n+1}$

If f is a measurable function for which $\mathrm{E}\left[\left|X_1\right|\right] < +\infty$, and if $Y = g\left(X\right)$ is bounded n-symmetric r.v., then for $1 \le j \le n$,

$$\begin{aligned}\mathrm{E}\left\{f\left(X_j\right)g\left(X\right)\right\} &= \mathrm{E}\left\{f\left(X_1\right)g\left(X_j, X_2, \cdots, X_{j-1}, X_1, X_{j+1}, \cdots\right)\right\} \\ &= \mathrm{E}\left\{f\left(X_1\right)g\left(X\right)\right\},\end{aligned}$$

so that
$$\mathrm{E}\left\{\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right)Y\right\} = \mathrm{E}\left\{f\left(X_1\right)Y\right\}.$$

(*continued*)

LLN for exchangeable sequences

proof : (*continued*)

Then, take Y as the indicator of A, 1_A, so that

$$\int_A \frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right) dP = \int_A f\left(X_1\right) dP \qquad \left(A \in \mathscr{F}_n\right).$$

Then, by the definition of conditional expectation,

$$\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right) = \mathrm{E}\left\{ f\left(X_1\right) \mid \mathscr{F}_n \right\}.$$

Since partial sums form a martingale, by the Martingale convergence theorem,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right) = \mathrm{E}\left\{ f\left(X_1\right) \mid \mathscr{F}_\infty \right\} \quad a.s. \quad \text{where } \mathscr{F}_\infty = \bigcap_{n=1}^{\infty} \mathscr{F}_n. \ \square$$

De Finetti's Theorem

Definition :

Let $\{X_i\}$ be r.v.s and let $\mathscr{F}$ be a σ-field.

Say $\{X_i\}$ is *conditionally i.i.d.* given $\mathscr{F}$ if for $A_i \subset \Re$

$$P\left(X_i \in A_i,\ 1 \le i \le n \mid \mathscr{F}\right) = \prod_i P\left(X_i \in A_i \mid \mathscr{F}\right), \text{ and}$$

$$P\left(X_i \in A \mid \mathscr{F}\right) = P\left(X_j \in A \mid \mathscr{F}\right) \text{ a.s., for each } A,\ i \ne j.$$

De Finetti’s Theorem

De Finetti's Theorem :

If $\{X_i\}$ is an exchangeable sequence then $\{X_i\}$ is conditionally i.i.d. given $\mathscr{F}_\infty$

Proof : By the LLN of exchangeable sequence,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right)=\mathrm{E}\left\{f\left(X_1\right)|\,\mathscr{F}_\infty\right\} \quad a.s. \quad \text{where } \mathscr{F}_\infty=\bigcap_{n=1}^{\infty}\mathscr{F}_n.$$

Let $f\left(y\right)=\begin{cases}1 & y\le x\\ 0 & y>x\end{cases}$. Then, $\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right)=\frac{1}{n}\#\left\{j\le n; X_j\le x\right\}$

and $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left(X_j\right)=\mathrm{E}\left\{f\left(X_1\right)|\,\mathscr{F}_\infty\right\}=\mathrm{P}\left(X_1\le x\,|\,\mathscr{F}_\infty\right)=\mathrm{F}\left(x\right)$,

where $\mathrm{F}\left(x\right)=\mathrm{P}\left\{X_1\le x\,|\,\mathscr{F}_\infty\right\}$ is a random distribution function. (*cont.*)

De Finetti's Theorem

Proof : (*cont.*) $X_1, \cdots X_n$ are i.i.d. $\Leftrightarrow$ $\exists$ F distribution function s.t.

$$\mathrm{P}\left(X_1 \le x_1 \wedge \cdots \wedge X_n \le x_n\right) = \mathrm{F}\left(x_1\right) \cdot \cdots \cdot \mathrm{F}\left(x_n\right) \quad \forall x_1, \cdots, x_n .$$

Since $\mathrm{P}\left(X_1 \le x_1\right) = \mathrm{E}\left\{I_{(-\infty, x_1]}\left(X_1\right)\right\}$,

$$\begin{aligned}\mathrm{P}\left(X_1 \le x_1 \wedge \cdots \wedge X_k \le x_k \mid \Im_\infty\right) &= \mathrm{E}\left\{I_{(-\infty, x_1]}\left(X_1\right) \cdot \cdots \cdot I_{(-\infty, x_k]}\left(X_k\right) \mid \mathscr{F}_\infty\right\} \\ &= \mathrm{F}\left(x_1\right) \cdot \cdots \cdot \mathrm{F}\left(x_k\right)\end{aligned}$$

$\forall x_1, \cdots, x_k$: $\omega \mapsto \mathrm{F}\left(\omega, x_1\right) \cdot \cdots \cdot \mathrm{F}\left(\omega, x_k\right)$ is $\mathscr{F}_\infty$-measurable.

$$\mathrm{E}\left\{\mathrm{E}\left\{I_{(-\infty, x_1]}\left(X_1\right) \cdot \cdots \cdot I_{(-\infty, x_k]}\left(X_k\right) \mid \mathscr{F}_\infty\right\} \mid F\right\} = E\left\{\mathrm{F}\left(x_1\right) \cdot \cdots \cdot \mathrm{F}\left(x_k\right) \mid F\right\}$$

where $F \subset \mathscr{F}_\infty$, and $\mathrm{F}\left(x_i\right)$ is F-measurable $\forall i = 1, 2, \cdots k$.

Then, $\quad \mathrm{E}\left\{I_{(-\infty, x_1]}\left(X_1\right) \cdot \cdots \cdot I_{(-\infty, x_k]}\left(X_k\right) \mid F\right\} = \mathrm{F}\left(x_1\right) \cdot \cdots \cdot \mathrm{F}\left(x_k\right)$,

and thus, $\quad \mathrm{P}\left(X_1 \le x_1 \wedge \cdots \wedge X_k \le x_k \mid F\right) = \mathrm{F}\left(x_1\right) \cdot \cdots \cdot \mathrm{F}\left(x_k\right)$. $\square$

Application to the mortgage mess

Let a r.v. X_i be the payoff from mortgage i and bank wants to spread the risk by making a large number of such mortgages and create a mortgage pool with deterministic payoff.

However, if X_i are not i.i.d but only exchangeable, there is some nontrivial σ-algebra $\mathscr{F}$ that underlies them all, i.e. X_i are conditionally i.i.d. on $\mathscr{F}$. Then the payoff seems to be (asymtotically) deterministic but is actually a random variable, $\mathscr{F}$-measurable, so its value changes depending on which set in $\mathscr{F}$, the event ω is in.

Thus, there are only exchangeable sequences in reality, there is no such a thing as i.i.d. sequence since there are always some underlying assumptions for which set $S \in \mathscr{F}$ we have $\omega \in S$ that can change .

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