

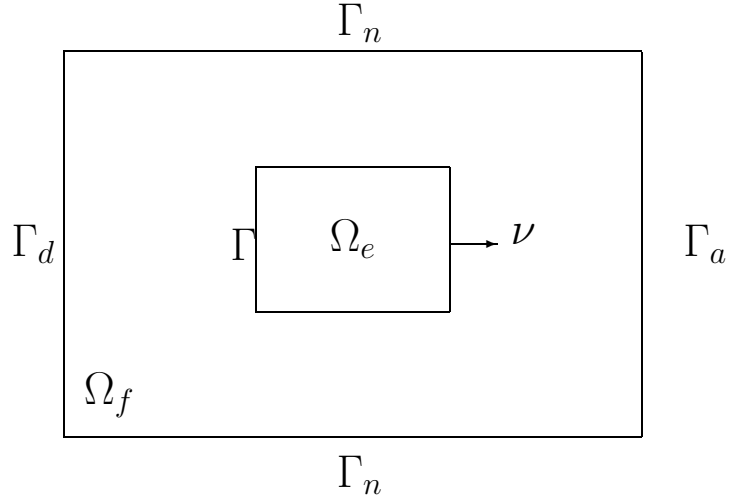
An Iterative Substructuring Method for Coupled Fluid-Solid Acoustic Problems

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Model problem



Pressure-displacement formulation

$$\begin{aligned} \Delta p + k^2 p &= 0 \text{ in } \Omega_f, \\ p &= p_0 \text{ on } \Gamma_d, \quad \frac{\partial p}{\partial \nu} = 0 \text{ on } \Gamma_n, \quad \frac{\partial p}{\partial \nu} + ikp = 0 \text{ on } \Gamma_a. \\ \nabla \cdot \tau + \omega^2 \rho_e u &= 0 \text{ in } \Omega_e, \\ \tau &= \lambda I(\nabla \cdot u) + 2\mu e(u), \quad e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \\ \nu \cdot u &= \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial \nu}, \quad \nu \cdot \tau \cdot \nu = -p, \quad \nu \times \tau \cdot \nu = 0 \text{ on } \Gamma \end{aligned}$$

Variational form and discretization

Define the spaces

$$V_f = \{q \in H^1(\Omega_f) | q = 0 \text{ on } \Gamma_d\}$$

$$V_e = \{u \in (H^1(\Omega_e))^n\}$$

Variational form: Find $p, p - p_0 \in V_f$ and $u \in V_e$ such that for all $q \in V_f$ and $v \in V_e$,

$$- \int_{\Omega_f} \nabla p \nabla q + k^2 \int_{\Omega_f} p q - ik \int_{\Gamma_a} p q - \int_{\Gamma} \rho_f \omega^2 (\nu \cdot u) q = 0$$

$$- \int_{\Omega_e} \lambda (\nabla \cdot u) (\nabla \cdot v) + 2\mu e(u) : e(v) + \omega^2 \int_{\Omega_e} \rho_e u \cdot v - \int_{\Gamma} p (\nu \cdot v) = 0$$

This variational formulation and discretization by conforming elements are well known (Morand and Ohayon 1995).

Discretized system

$$\begin{bmatrix} -\mathbf{K}_f + k^2\mathbf{M}_f - ik\mathbf{G}_f & -\rho_f\omega^2\mathbf{T} \\ -\mathbf{T}^* & -\mathbf{K}_e + \omega^2\mathbf{M}_e \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix}$$

$$\mathbf{p}^*\mathbf{K}_f\mathbf{q} = \int_{\Omega_f} \nabla p \nabla q,$$

$$\mathbf{p}^*\mathbf{M}_f\mathbf{q} = \int_{\Omega_f} pq,$$

$$\mathbf{p}^*\mathbf{G}_f\mathbf{q} = \int_{\Gamma_a} pq,$$

$$\mathbf{u}^*\mathbf{K}_e\mathbf{v} = \int_{\Omega_e} \lambda(\nabla \cdot u)(\nabla \cdot v) + 2\mu e(u) : e(v),$$

$$\mathbf{u}^*\mathbf{M}_e\mathbf{v} = \omega^2 \int_{\Omega_e} \rho_e u \cdot v,$$

$$\mathbf{p}^*\mathbf{T}\mathbf{v} = \int_{\Gamma} p(\nu \cdot v).$$

Decomposition

Non-overlapping subdomains:

$$\bar{\Omega}_f = \bigcup_{s=1}^{N_f} \bar{\Omega}_e^s, \quad \bar{\Omega}_e = \bigcup_{s=1}^{N_e} \bar{\Omega}_e^s.$$

Vector of all subdomain dofs:

$$\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p}^1 \\ \vdots \\ \mathbf{p}^{N_f} \end{bmatrix}, \quad \hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^{N_e} \end{bmatrix},$$

Corresponding partitioned matrices with subdomain blocks defined by subassembly,

$$\hat{\mathbf{K}}_f = \begin{bmatrix} \mathbf{K}_f^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{K}_f^{N_f} \end{bmatrix}, \quad \mathbf{p}^{\mathbf{s}*} \mathbf{K}_f^s \mathbf{q} = \int_{\Omega_f^s} \nabla p \nabla q,$$

($\hat{\mathbf{K}}_e$ $\hat{\mathbf{M}}_f$ $\hat{\mathbf{M}}_e$ defined similarly)

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{T}^{11} & \dots & \mathbf{T}^{1,N_e} \\ \vdots & \ddots & \vdots \\ \mathbf{T}^{N_f,1} & \dots & \mathbf{T}^{N_f,N_e} \end{bmatrix}, \quad \mathbf{p}^{\mathbf{r}*} \mathbf{T}^{rs} \mathbf{v}^s = \int_{\partial\Omega_f^r \cap \partial\Omega_e^s} p(\nu \cdot v)$$

Intersubdomain continuity

Local to global maps \mathbf{N}_f and \mathbf{N}_e :

$$\mathbf{K}_f = \mathbf{N}_f^* \hat{\mathbf{K}}_f \mathbf{N}_f, \quad \mathbf{K}_e = \mathbf{N}_e^* \hat{\mathbf{K}}_e \mathbf{N}_e$$

$$\hat{\mathbf{p}} = \mathbf{N}_f \mathbf{p}, \quad \hat{\mathbf{u}} = \mathbf{N}_e \mathbf{u}.$$

To enforce same values between subdomains:

$$\mathbf{B}_f = (\mathbf{B}_f^1, \dots, \mathbf{B}_f^{N_f}), \quad \mathbf{B}_e = (\mathbf{B}_e^1, \dots, \mathbf{B}_e^{N_e})$$

such that

$$\mathbf{B}_f \hat{\mathbf{p}} = 0 \iff \hat{\mathbf{p}} = \mathbf{N}_f \mathbf{p} \text{ for some } \mathbf{p}$$

$$\mathbf{B}_e \hat{\mathbf{u}} = 0 \iff \hat{\mathbf{u}} = \mathbf{N}_e \mathbf{u} \text{ for some } \mathbf{u}.$$

Decomposed system

$$\begin{bmatrix} -\hat{\mathbf{K}}_f + k^2 \hat{\mathbf{M}}_f & -\omega^2 \rho_f \hat{\mathbf{T}} & \mathbf{B}_f^* & 0 \\ -\hat{\mathbf{T}}^* & -\hat{\mathbf{K}}_e + \omega^2 \hat{\mathbf{M}}_e & 0 & \mathbf{B}_e^* \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$(\hat{\mathbf{p}}, \hat{\mathbf{u}}, \lambda_f, \lambda_e)$ equivalent to the original system via $\hat{\mathbf{p}} = \mathbf{N}_f \mathbf{p}$ and $\hat{\mathbf{u}} = \mathbf{N}_e \mathbf{u}$.

Regularized system

$$\begin{bmatrix} \hat{\mathbf{A}}_f & -\omega^2 \rho_f \hat{\mathbf{T}} & \mathbf{B}_f^* & 0 \\ -\hat{\mathbf{T}}^* & \hat{\mathbf{A}}_e & 0 & \mathbf{B}_e^* \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where

$$\begin{aligned} \hat{\mathbf{A}}_f &= -\hat{\mathbf{K}}_f + k^2 \hat{\mathbf{M}}_f + \hat{\mathbf{R}}_f \\ \hat{\mathbf{A}}_e &= -\hat{\mathbf{K}}_e + \omega^2 \hat{\mathbf{M}}_e + \hat{\mathbf{R}}_e \end{aligned}$$

$$\begin{aligned} \hat{\mathbf{R}}_f &= (\mathbf{R}_f^{rs})_{rs}, & \mathbf{p}^{\mathbf{s}*} \mathbf{R}_f \mathbf{q}^{\mathbf{s}} &= ik \sum_{t \neq s} \sigma_{st} \int_{\partial \Omega_f^s \cap \partial \Omega_f^t} pq, \\ \hat{\mathbf{R}}_e &= (\mathbf{R}_e^{rs})_{rs}, & \mathbf{u}^{\mathbf{s}*} \mathbf{R}_e \mathbf{v}^{\mathbf{s}} &= i\omega \sum_{t \neq s} \sigma_{st} \int_{\partial \Omega_e^s \cap \partial \Omega_e^t} (\mathbf{n} \cdot \mathbf{u})(\mathbf{n} \cdot \mathbf{v}), \end{aligned}$$

where $\sigma_{st} = \pm 1$, $\sigma_{st} = -\sigma_{ts}$.

If σ_{st} does not change sign on Ω_f^s , then $-\hat{\mathbf{K}}_f^s + k^2 \hat{\mathbf{M}}_f^s + \hat{\mathbf{R}}_f^r$ is regular (Farhat, Macedo, Lesoinne 2000). Similarity for solid subdomains.

In computational tests, we assigned σ_{st} by counting, did not try to avoid change of sign.

Augmented system

Key: $\hat{\mathbf{T}}\hat{\mathbf{u}}$, $\hat{\mathbf{T}}^*\hat{\mathbf{p}}$ depend on the values of $\hat{\mathbf{u}}$, $\hat{\mathbf{p}}$ on the wet interface Γ only. Define $\hat{\mathbf{J}}_f, \hat{\mathbf{J}}_e$ as expanding vector on Γ by zero entries, then

$$\hat{\mathbf{T}}\hat{\mathbf{u}} = \hat{\mathbf{T}}\mathbf{J}_e\hat{\mathbf{u}}_\Gamma, \quad \hat{\mathbf{u}}_\Gamma = \mathbf{J}_e^*\hat{\mathbf{u}},$$

$$\hat{\mathbf{T}}^*\hat{\mathbf{p}} = \hat{\mathbf{T}}^*\mathbf{J}_f\hat{\mathbf{p}}_\Gamma, \quad \hat{\mathbf{p}}_\Gamma = \mathbf{J}_f^*\hat{\mathbf{p}}.$$

Get the augmented system

$$\begin{bmatrix} \hat{\mathbf{A}}_f & 0 & \mathbf{B}_f^* & 0 & 0 & -\omega^2\rho_f\hat{\mathbf{T}}\mathbf{J}_e \\ 0 & \hat{\mathbf{A}}_e & 0 & \mathbf{B}_e^* & -\hat{\mathbf{T}}^*\mathbf{J}_f & 0 \\ \mathbf{B}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 & 0 & 0 \\ -\mathbf{J}_f^* & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & -\mathbf{J}_e^* & 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \\ \hat{\mathbf{p}}_\Gamma \\ \hat{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Proposed method

- eliminate $\hat{\mathbf{p}}$ and $\hat{\mathbf{u}}$ from the augmented system
- solve the resulting reduced system by Generalized Conjugate Residuals
- precondition by scaling and projection on a coarse space defined by rigid body modes and plane waves

Scaling

- multiply the second equation by $\omega^2 \rho_f$
- symmetric diagonal scaling

$$\begin{bmatrix} \tilde{\mathbf{A}}_f & 0 & \tilde{\mathbf{B}}_f^* & 0 & 0 & -\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & \tilde{\mathbf{A}}_e & 0 & \tilde{\mathbf{B}}_e^* & -\tilde{\mathbf{T}}^*\mathbf{J}_f & 0 \\ \tilde{\mathbf{B}}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{B}}_e & 0 & 0 & 0 & 0 \\ -\mathbf{J}_f^* & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & -\mathbf{J}_e^* & 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}} \\ \tilde{\mathbf{u}} \\ \tilde{\lambda}_f \\ \tilde{\lambda}_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} \tilde{\mathbf{A}}_f &= \mathbf{D}_f \hat{\mathbf{A}}_f \mathbf{D}_f, & \tilde{\mathbf{A}}_e &= \omega^2 \rho_f \mathbf{D}_e \hat{\mathbf{A}}_e \mathbf{D}_e, & \tilde{\mathbf{T}} &= \omega^2 \rho_f \mathbf{D}_f \hat{\mathbf{T}} \mathbf{D}_e, \\ \tilde{\mathbf{B}}_f &= \mathbf{E}_f \mathbf{B}_f \mathbf{D}_f, & \tilde{\mathbf{B}}_e &= \mathbf{E}_e \mathbf{B}_e \mathbf{D}_e, & \tilde{\mathbf{r}} &= \mathbf{D}_f \hat{\mathbf{r}}, \\ \tilde{\mathbf{p}} &= \mathbf{D}_f \hat{\mathbf{p}}, & \tilde{\mathbf{u}} &= \mathbf{D}_e \hat{\mathbf{u}}, & \lambda_f &= \mathbf{D}_f \tilde{\lambda}_f, & \lambda_e &= \omega^2 \rho_f \mathbf{D}_f \tilde{\lambda}_f. \end{aligned}$$

Reduced system

Eliminating

$$\begin{aligned}\tilde{\mathbf{p}} &= \tilde{\mathbf{A}}_f^{-1}(\tilde{\mathbf{r}} - \tilde{\mathbf{B}}_f^* \tilde{\lambda}_f + \tilde{\mathbf{T}} \mathbf{J}_e \tilde{\mathbf{u}}_\Gamma) \\ \tilde{\mathbf{u}} &= \tilde{\mathbf{A}}_e^{-1}(-\tilde{\mathbf{B}}_e^* \tilde{\lambda}_e + \tilde{\mathbf{T}}^* \mathbf{J}_f \tilde{\mathbf{p}}_\Gamma)\end{aligned}$$

gives

$$\mathbf{F} \mathbf{x} = \mathbf{b},$$

where

$$\mathbf{F} = \begin{bmatrix} \tilde{\mathbf{B}}_f \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{B}}_f^* & 0 & 0 & -\tilde{\mathbf{B}}_f \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{T}} \mathbf{J}_e \\ 0 & \tilde{\mathbf{B}}_e \tilde{\mathbf{A}}_e^{-1} \tilde{\mathbf{B}}_e & -\tilde{\mathbf{B}}_e \tilde{\mathbf{A}}_e^{-1} \tilde{\mathbf{T}}^* \mathbf{J}_f & \\ \mathbf{J}_f^* \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{B}}_f^* & 0 & \mathbf{I} & \mathbf{J}_f^* \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{T}} \mathbf{J}_e \\ 0 & \mathbf{J}_e \tilde{\mathbf{A}}_e^{-1} \tilde{\mathbf{B}}_e & -\mathbf{J}_e \tilde{\mathbf{A}}_e^{-1} \tilde{\mathbf{T}}^* \mathbf{J}_f & \mathbf{I} \end{bmatrix},$$

and

$$\mathbf{x} = \begin{bmatrix} \lambda_f \\ \lambda_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \tilde{\mathbf{B}}_f \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{r}} \\ 0 \\ -\mathbf{J}_f^* \tilde{\mathbf{A}}_f^{-1} \tilde{\mathbf{r}} \\ 0 \end{bmatrix}.$$

First diagonal block is FETI-H operator, 2nd analogue for elasticity. Assuming FETI-H operator is well conditioned, the reduced system has the form of Fredholm integral equation of 2nd kind, hence well conditioned!

Iterative solution

Enforce the residual condition

$$\mathbf{Q}^*(\mathbf{F}\mathbf{x} - \mathbf{b}) = 0$$

throughout the iterations. For given \mathbf{v} use the initial approximation

$$\mathbf{x}^{(0)} = \mathbf{v} + \mathbf{Q}\mathbf{w},$$

\mathbf{w} obtained by solving the residual correction equation,

$$\mathbf{Q}^*(\mathbf{F}(\mathbf{v} + \mathbf{Q}\mathbf{w}) - \mathbf{b}) = 0.$$

$\mathbf{F}\mathbf{x} = \mathbf{b}$ solved by GCR with left preconditioning by the projection

$$\mathbf{P} = \mathbf{I} - \mathbf{Q}(\mathbf{Q}^*\mathbf{F}\mathbf{Q})^{-1}\mathbf{Q}^*\mathbf{F}$$

and initial iterate $\mathbf{x}^{(0)}$. Equivalently, GCR applied to

$$\mathbf{P}\mathbf{F}\mathbf{x} = \mathbf{P}\mathbf{b}.$$

Iterations run in a subspace:

$$\mathbf{Q}^*\mathbf{F}(\mathbf{x}^{(n)} - \mathbf{x}^{(0)}) = 0$$

Selection of coarse space

$$\mathbf{Q} = \begin{bmatrix} \mathbf{D}_f \mathbf{B}_f \mathbf{Y}_f & 0 & 0 & 0 \\ 0 & \mathbf{D}_e \mathbf{B}_e \mathbf{Y}_e & 0 & 0 \\ 0 & 0 & \mathbf{D}_f \mathbf{J}_f^* \mathbf{Z}_f & 0 \\ 0 & 0 & 0 & \mathbf{D}_f \mathbf{J}_f^* \mathbf{Z}_e \end{bmatrix}$$

Coarse selection for multipliers:

$\mathbf{Y}_f = \text{diag}(\mathbf{Y}_f^s)$, columns of \mathbf{Y}_f^s are discrete representations of plane waves in a small number of equally distributed directions, or discrete representation of the constant function.

$\mathbf{Y}_e = \text{diag}(\mathbf{Y}_e^s)$, columns of \mathbf{Y}_e^s are discrete representations of elastic plane waves (both pressure and shear) in a small number of equally distributed directions, or discrete representation of the rigid body motions

Coarse selection for wet interface:

The matrices \mathbf{Z}_f^s and \mathbf{Z}_e^s are chosen in the same way as \mathbf{Y}_f^s and \mathbf{Y}_e^s , with possibly different selection of the number of directions and selection of constant or rigid body modes.

Some of the matrices \mathbf{Y}_f^s , \mathbf{Y}_e^s , \mathbf{Z}_e^s , or \mathbf{Y}_e^s may be void.

Conclusion

- For stiff heavy scatterer, fluid and solid decouple in the limit
- Reduces to FETI-H (Farhat et al, 1998, 2000) if scatterer immovable
- 2-3x more iterations than FETI-H if all else same, scalable
- some hope for theoretical justification – 2nd kind Fredholm equations?
- fluid and solid discretizations do not have to match (but they do in the model problem)
- Matlab prototype implementation

Future work

- implementation: large problems, 3D, parallel,...
- nonreflecting far field boundary condition
- realistic elastic body: plates, shells, junctures
How to choose coarse space in the solid?