Math 5663 Fall 2004
Introduction to Finite Element Methods
Classroom notes and homeworks

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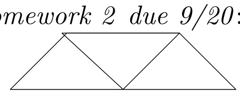
- 8/23/04: Variational formulation of 1D BVP (Johnson 1.1), derivation of truss equation (handout) *Homework 1 due 8/30*: Let g be continuous on $(0,1)$, $g(0) = g(1) = 0$, $\int_0^1 gvdx = 0 \forall v : v$ continuous on $(0,1)$, $v(0) = v(1) = 0$. Then $g = 0$ on $(0,1)$. Note: the same solution applies when $(0,1)$ is replaced by $[0,1]$ or when the zero boundary values are omitted.

- 8/25/04: Finite element approximation in 1D (Johnson 1.2)

- 8/30/04: Basic error analysis (Johnson 1.3), abstract function space formulation (Johnson 2)

- 9/1/04: Dirac delta, Green's function, FEM exact in 1D (Hughes p. 27)

- 9/8/04: Local stiffness matrix, assembly (Hughes 2.5, 2.6)

- 9/13/04: the truss (spring) element, direct stiffness method in nD (handout: `element_truss.m`, boards photos), balance of forces and assembly for elasticity (Hughes 2.10) in the case of trusses. Essential boundary conditions in 1D. *Homework 2 due 9/20*: consider a bridge model as in the following picture.  The bridge is made of trusses from aluminum and crossection 5cm^2. The height of the bridge is 1m; width of each of the bottom parts is 2m. The nodes at the ends of the bottom are fixed. The load at the middle node on the bottom is 1000N in the down direction. Find the displacement of that node. Use files in the directory matlab/bridge.

- 9/15/04: constraints: non-homogeneous essential boundary conditions and natural boundary conditions in 1D (board photos)

- 9/20/04: finite elements for the Poisson equation in 2D (Johnson 1.4-1.7). *Homework 3 due 9/27*:

1. Write a function that implements a triangular element for the Laplace equation:

```
function out=element_tria3(task,nodes,coef)
% K=element_tria3('stiff',nodes,coef)
% 3-noded triangle
% input:
%   task    'stiff'
%   nodes   node(:,i) are xyz coordinates of node i
%   coef    the coefficient
% output:
%   K       local stiffness matrix
```

   The z coordinate can be assumed to be always zero.

2. Implement a test problems to validate your element: For the square $[0,1] \times [0,1]$ divided into 4 triangles that meet at the point $[0.4, 0.6]$ create the right hand side for the Neumann boundary condition satified by the exact solution $u(x,y) = a*x + b*y + c$ and verify that the exact solution satifies the discrete equations. Choose a, b, c as random. Use `assemble.m` to get the global stiffness matrix.

- 9/22/04: details on Neumann problem (Johnson 1.7) and its implementation (board photos); continue Johnson 1.4

- 9/27/04: Sobolev spaces (Johnson 1.5), numerical quadrature on triangles (handout, boards). *Homework 4 due 10/4/04:*

1. Write a function to create a triangular mesh with $2mn$ elements obtained by dividing each rectangle in two (fig. 1.10, Johnson p. 31) in an $m\times$ n rectangular mesh with spacing h_1 and h_2 in the x and y direction, respectively. Verify your mesh and the element using your validation code from Homework 3.2 for $m = 3$, $n = 2$, $h_1 = 0.4$, $h_2 = 1$.

2. Write a function `u=dirichlet(m,n,f,order)` to solve the Dirichlet problem

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega = [0,1] \times [0,1] \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

   on your mesh from problem 1. Here `order=1` or `2` is the degree of polynomials integrated exactly by the quadrature formula for the right-hand side and `f` is a function handle.

3. Verify your code for the case $f(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$ by tabulating the difference $d(m,n)$ between the numerical solution and the exact solution (compute it!) at the point $[1/2, 1/2]$ for m, $n = 2, 4, 8, 16, 32$ for both quadrature orders. For $m = n$, the error is expected to behave as $d(m,n) \approx h^{-\alpha}$, $h = 1/n$; determine α.

- 9/29/04: The space L^2 as completion, weak derivative, Sobolev spaces H^1 and H_0^1 (Johnson 1.5, boards)

- 10/4/04: Substitution in multidimensional integrals, mapping from a reference element, numerical quadrature on the reference element by tensor product Gaussian quadrature

- 10/6/04: Computing the local stiffness matrix (handout)

- 10/11/04: isoparametric element, the bilinear quadrilateral isoparametric (quad4) element (Hughes 3.1-3.3, Johnson 12.1-12.2). *Homework 5 due 10/25/04*:

  1. Write a function `element_quad4` that implements the quad4 element using 4 node Gaussian quadrature on the reference square, with the same interface as in homework 3.1
  2. Implement in a function `element_quad4_test` a test problem to validate your element: For the square $[0, 1] \times [0, 1]$ divided into 4 rectangles with vertices at the midpoint on the sides and at the point $[0.4, 0.6]$, create the right hand side for the Neumann boundary condition satified by the exact solution $u(x, y) = a*x+b*y+c$ and verify that the exact solution satifies the discrete equations. Choose a, b, c as random. Use `assemble.m` to get the global stiffness matrix. The function returns the residual in the discrete equations, which should be zero up to rounding.
  3. Write a function `u=dirichlet_quad4(m,n,f)`that solves the same problem as in Homework 4.2 but uses the quad4 elements instead of the tria3 element.
  4. Create the tables of 1. the difference between the exact and the numerical error 2. the ratios of the error for the tria3 element and the error for the quad4 element, for $m = n = 1/2, 1/4, 1/8, 1/16, 1/32$

- 10/13/04: requirements for quadrature order in isoparametric elements (Johnson 12.2, Hughes, 3.11 exercise 7a; boards); linear response of the isoparametric quadrilateral

- 10/18/04: Midterm review

  1. Use Hooke's law and the direct stiffness method to derive the stiffness matrix of the 1D truss element.
  2. Derive formally the variational form of the boundary value problem $-u'' = f, u(0) = u(1) = 0$.
  3. Same in 2D (with the hint $\int\limits_{\Omega} \nabla u \cdot \nabla v dx + \int\limits_{\Omega} (\Delta u)\, v dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} v ds$).
  4. Derive formally the variational form of the boundary value problem $-u'' = f, u(0) = 0, u'(1) = 1$

5. Given variational form of this problem derive the discrete equations for the linear elements.
6. Derive formally the variational form of the boundary value problem $-u'' = f, u(0) = 0, u(1) = 1$
7. State and derive the basic error estimate in energy norm in terms of best approximation (Cea's lemma)
8. Compute the finite element discretization of of the 1D problems above with $h = 1/2$.
9. Given a basis of $\{\varphi_i\}$ of V_h derive the matrix form of the discrete variational problem $u \in V_h, a(u, v) = F(v) \forall v \in V_h$
10. Given a 1D problem as above and two elements compute the local matrices and the local right-hand sides and assemble the global problem.
11. Consider the problem $-u'' = x^4$ solved by linear elements, how many nodes of Gaussian quadrature are needed for the solution to be exact? Justify.
12. State the conditions for a function defined on each element separately to be in H^1
13. State the definition of a weak derivative and show that if a function has the classical derivative then it is also the weak derivative.
14. Show that the jump function does not have a weak derivative that would be a function. It is the Dirac delta "function".
15. Describe the shape functions of the triangular linear element.
16. Describe the shape functions of the isoparametric quadrilateral element. When are the shape functions polynomial?
17. Describe computation of the local stiffness matrix for isoparametric elements.
18. Describe the shape functions for the 1D linear element.

- 10/20/04: Midterm exam
- 10/25/04: Some finite element spaces (Johnson ch. 3): isoparametric coordinates, quadratic triangular element. *Homework 6 due 11/1/04*:
  1. Johnson page 82 problem 3.4
  2. problem 3.6
- 10/27/04: Johnson ch. 3 - higher order elements: cubic triangles.
- 11/1/04: Johnson ch. 3 - linear tetrahedron, C^1 elements. *Homework 7 due 11/8/04:*
  1. Johnson page 82 problem 3.7

2. Show that the functions in the finite element space corresponding to problem 3.7 are in C^1, or provide a counterexample.

- 11/3/04: Elasticity: stress tensor
- 11/8/04: Elasticity: strain tensor, generalized Hooke's law *Homework 8 due 11/15/04:* write the isotropic material stress-strain relationship $\tau_{ij} = \lambda\delta_{ij} \operatorname{div} u + 2\mu e_{ij}$ in the form $\tau_{ij} = c_{ijkl}e_{kl}$ (that is, find c_{ijkl} as functions of λ and μ)
- 11/10/04: variational formulation or linear elasticity (Hughes Sec. 2.7), Korn's inequality
- 11/15/04: strain tensor review, plane strain, plane strain, plane stress Cea's lemma (general case, Johnson 2.2) *Homework 9 due 11/29/04:*

1. Write a function `element_quad4_plane` that implements the quad4 element for plane elasticity using 4 node Gaussian quadrature on the reference square. Interface:

```
function out=element_quad4_plane(task,nodes,coef)
% K=element_quad4_plane('stiff',nodes,coef)
% 4-node plane elasticity
% input:
%   task    'stiff'
%   nodes   node(:,i) are xyz coordinates of node i
%   coef    the coefficients, dimension [2,2,2,2]
% output:
%   K       local stiffness matrix
```

   Verify that the local stiffness matrix has exactly 3 zero eigenvalues.

2. Implement in a function `element_quad4_plane_test` a test problem to validate your element: For the square $[0, 1] \times [0, 1]$ divided into 4 rectangles with vertices at the midpoint on the sides and at the point $[0.4, 0.6]$, create the right hand side for the traction boundary condition satified by the exact solution $u(x) = Ax$, where A is a 2×2 matrix, and verify that the exact solution satifies the discrete equations. Choose A as random. Use `assemble.m` to get the global stiffness matrix. The function should returns the residual in the discrete equations, which should be zero up to rounding.

- 11/17/04: Lax-Milgram theorem (Johnson 2.1), correspondence of bilinear forms and linear operators
- 11/29/04: Correspondence of minimization and variational form, constrained minimization, almost incompressible elasticity. *Homework 10 due 12/6/04*:

1. Let A be n by n symmetric positive definite matrix, f an n vector, and define
$$J(u) = \frac{1}{2}u^T Au - u^T f$$
Prove that
$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \nabla J(u) \cdot v = (Au - f)^T v$$
2. Let $c(\cdot, \cdot)$ be a symmetric, positive definite bilinear form on a linear space V (i.e., c linear in each argument separately, $c(u, u) > 0 \forall u \in V$, and $c(u, v) = c(v, u) \forall u, v \in V$) and f a linear functional on V. Define
$$J(u) = \frac{1}{2}c(u, u) - f(u)$$
and prove that
$$[J(w) \geq J(u) \forall w \in V] \Leftrightarrow [c(u, v) = f(v) \forall v \in V]$$

- 12/1/04: almost incompressible locking, Stokes problem and divergence free functions (Johnson 5.2), mixed elements
- 12/6/04: inf-sup condition for the Stokes problem. Computer demonstration of Homework 9 and incompressible locking. Concluding presentation "When Point Boundary Conditions Are Meaningful and When They Are Not, or, Why We Need Functional Analysis" (see find constr04.pdf in handouts). Student course evaluations.
- 12/8/04: Review for final:

1. Describe shape functions on linear triangular element using barycentric coordinates.
2. Define the property of an element to be unisolvent. [for every vector of values c_i of the degrees of freedom d_i on the element there exists unique function v from the element space such that the values of the degrees of freedom equal to the prescribed values: $d_i(v) = c_i$] How does one verify that an element is unisolvent?
3. Prove that if the only function in the element space with all degrees of freedom equal to zero is the zero function, then if two functions from the finite element space have the same values of degrees of freedom, the functions equal.
4. What do you need to show to prove that an element is C^0? [answer: values on each edge (face in 3D) are determined only by degrees of freedom on that edge (face) and they coincide for two elements that share the edge (face)] and C^1[the same also for normal derivatives]. Why not for tangential derivatives? [because the tangential derivatives coincide in any case]

5. Give an example of a C^1 element (formulation, do not prove anything)
6. Define the stress tensor in terms of force on a small surface inside the material, and use it to obtain the balance of forces equations. [answer: $F = \tau\nu dA$, τ a matrix, $\tau_{ij,i} = 0$ in tensor notation or $\sum_i \frac{\partial \tau_{ij}}{\partial x_i} = 0 \forall j$ in classical notation, or $\operatorname{div} \tau = 0$,get it by applying Stokes theorem to $u_i = \tau_{ij}$ for fixed i]
7. Define the small strain tensor, the generalized Hooke's law (stress-strain relationship), the balance of forces equation, and write the elasticity equations as partial differential equations for the displacement u. [answer: $u_{(i,j)}$, $\tau_{ij} = c_{ijkl} u_{(k,l)}$, $\left(c_{ijkl} u_{(k,l)}\right)_i + f_j = 0$]
8. What are the traction boundary conditions in elasticity? [answer: $\tau_{ij}\nu_i = g_j$] . Explain.
9. Write the variational form of the linearized elastostatic equations with homogeneous Dirichlet boundary conditions (see Nov. 10 lecture)
10. What symmetries are satisfied by the stress tensor τ_{ij}, the small strain tensor $u_{(i,j)}$, and the elasticity coefficients c_{ijkl} ?
11. Formulate the V - ellipticity condition for elasticity, how is it called [Korn's inequality]
12. What are rigid body modes, characterize in terms of small strain tensor (no proof)
13. Formulate the assumptions of the plane strain and plane stress problem
14. Formulate the Lax-Milgram theorem
15. Formulate the conditions for optimality of $J(u) = \frac{1}{2} u^T A u - u^T f$ subject to $Bu = g$ using Lagrange multipliers; for simplicity, work in finite dimension and in terms of matrices.
16. Derive the Stokes problem as limiting case of almost incompressible elasticity (substitution $p = \lambda \operatorname{div} u$)
17. Formulate and explain the inf-sup condition for the Stokes problem
18. Formulate and explain the discretization error estimate for the Stokes problem and relate to the inf-sup condition (the inf-sup condition forces V_h to be large; but when V_h is too large the increase does not contribute to accuracy which is limited by the approximation of p in Q_h)

- 12/13/04: no class, office hours only
- 12/15/04 (Wednesday): Final 5-6:15PM