

# 1 Integrating the stiffness matrix

Index after comma mean differentiation. Repeated index implies summation.

## 1.1 Laplace equation

The basis functions  $f$  are defined on a reference element  $R$  with coordinates  $\xi_j$ . The functions  $x = x(\xi)$ ,  $x = [x_i]$  define the element  $E$  as the image  $x(R)$ . Denote

$$x_{i,j} = \frac{dx_i}{d\xi_j}$$

Basis functions  $g$  on  $E$  are defined via this mapping:

$$g_a(x) = f_a(\xi)$$

where  $a$  is the index of the basis function. We wish to compute the entries of the local stiffness matrix of element  $E$ , given by

$$K_{ab} = \int_E c g_{a,j} g_{b,j} dx \quad (1)$$

where  $a$  and  $b$  are the numbers of the shape functions (or nodes),  $j$  is the index of the partial derivatives, and  $c > 0$  is constant coefficient.

Using the substitution  $x = x(\xi)$  and the chain rule  $g_{a,j}(x) = f_{a,p}(\xi) \xi_{p,j}(x)$  we get from (2)

$$K_{ab} = \int_R v_{ab}(\xi) d\xi$$

where

$$v_{ab} = c f_{a,p} \xi_{p,j} f_{b,q} \xi_{q,j} \det [x_{m,n}]$$

We do not know the partial derivatives

$$\xi_{p,j} = \frac{\partial \xi_p}{\partial x_j}$$

but we are given the mapping  $x = x(\xi)$  so  $\xi = \xi(x)$  is the inverse mapping and its partial derivatives are the entries of the inverse matrix to the Jacobian matrix of  $x$ ,

$$[\xi_{p,j}] = [x_{j,p}]^{-1}$$

To compute the stiffness matrix numerically we need the quadrature nodes  $z_n$  and weights  $w_n$  on the reference element  $R$ ,

$$\int_R v(\xi) d\xi \approx w_n v(z_n)$$

(again, with summation over the repeated index  $n$ ). In conclusion, we get the formula for evaluating the stiffness matrix

$$K_{ab} = w_n [c f_{a,p} \xi_{p,j} f_{b,q} \xi_{q,j} \det [x_{m,n}]]_{\xi=z_n} \quad \text{where} \quad [\xi_{p,j}] = [x_{j,p}]^{-1}$$

So, to evaluate the stiffness matrix we need: quadrature nodes  $z_n$  and weights  $w_n$  on the reference element  $R$ , partial derivatives  $f_{a,p}$  of the reference basis functions at the quadrature nodes, partial derivatives  $x_{m,n}$  of the mapping that defines the element  $E = x(R)$  and the coefficient  $c$ .

For quadrature of the right-hand side the situation is simpler: then we get

$$\begin{aligned} F_a &= \int_E F(x) g_a(x) dx \\ &= \int_R \underbrace{f(x(\xi)) f_a(\xi) \det [x_{m,n}]}_{u_a(\xi)} d\xi \\ &= w_n u_a(z_n) \end{aligned}$$

## 1.2 General formula

The basis functions  $f$  are defined on a reference element  $R$  with coordinates  $\xi_j$ . The functions  $x = x(\xi)$ ,  $x = [x_i]$  define the element  $E$  as the image  $x(R)$ . Note that

$$x_{i,j} = \frac{dx_i}{d\xi_j}$$

Basis functions  $g$  on  $E$  are defined via this mapping:

$$g_a(x) = f_a(\xi)$$

where  $a$  is the index of the basis function. We wish to compute the entries of the local stiffness matrix of element  $E$ , given by

$$K_{iakb} = \int_E c_{ijkl} g_{a,j} g_{b,l} dx \quad (2)$$

where  $a$  and  $b$  are the numbers of the shape functions (or nodes),  $j$  and  $l$  are the indices of the partial derivatives, and  $c'_{ijkl}$  are constant coefficients.

Note that the formula (2) is quite general: it works for elasticity where  $i$  and  $k$  are the directions of the displacement ( $c_{ijkl}$  are the same as the coefficients from the Hooke's law, see Sec. 1.3), and in applications where the range of the indices  $i$  and  $k$  is just 1 unrelated to the dimension of the space (Sec. 1.1). Using the substitution  $x = x(\xi)$  and the chain rule  $g_{a,j}(x) = f_{a,p}(\xi)\xi_{p,j}(x)$  we get from (2)

$$K_{iakb} = \int_R v_{iakb}(\xi) d\xi$$

where

$$v_{iakb} = c'_{ijkl} f_{a,p} \xi_{p,j} f_{b,q} \xi_{q,l} \det [x_{m,n}]$$

We do not know the partial derivatives

$$\xi_{p,j} = \frac{\partial \xi_p}{\partial x_j}$$

but we are given the mapping  $x = x(\xi)$  so  $\xi = \xi(x)$  is the inverse mapping so the partials are the entries of the inverse matrix

$$[\xi_{p,j}] = [x_{m,n}]^{-1}$$

To compute the stiffness matrix numerically we need the quadrature nodes  $z_n$  and weights  $w_n$  on the reference element  $R$ ,

$$\int_R v(\xi) d\xi \approx w_n v(z_n)$$

(again, with summation over the repeated index  $n$ ). In conclusion, we get the formula for evaluating the stiffness matrix

$$\begin{aligned} K_{iakb} &= w_n v_{iakb}(z_n) \\ v_{iakb} &= c'_{ijkl} f_{a,p} \xi_{p,j} f_{b,q} \xi_{q,l} \det [x_{m,n}] \\ [\xi_{p,j}] &= [x_{m,n}]^{-1} \end{aligned}$$

So, to evaluate the stiffness matrix we need: quadrature nodes  $z_n$  and weights  $w_n$  on the reference element  $R$ , partial derivatives  $f_{a,p}$  of the reference basis

functions at the quadrature nodes, partial derivatives  $x_{m,n}$  of the mapping that defines the element  $E = x(R)$  and the coefficients  $c'_{ijkl}$ .

Note that an efficient and stable way of evaluating determinant is as well as the inverse matrix are from a decomposition such as the QR decomposition:

$$A = QR$$

where  $Q$  is an orthogonal matrix and  $R$  is an upper triangular matrix. Then

$$\begin{aligned} A^{-1} &= R^{-1}Q^{-1} = R^{-1}Q^T \\ \det A &= \det Q \det R = \pm r_{11} \cdots r_{nn} \end{aligned}$$

See Numerical Analysis for details.

### 1.3 Elasticity

The bilinear form for elasticity is

$$a(u, v) = \int_{\Omega} u_{(i,j)} c_{ijkl} v_{(k,l)} dx$$

where the subscript in parenthesis means symmetric part:

$$u_{(i,j)} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

Because of the symmetry  $c_{ijkl} = c_{jikl}$  we have

$$u_{(i,j)} c_{ijkl} = u_{i,j} c_{ijkl}$$

and the stiffness matrix on element  $E$  can be obtained by integrating over just one element  $E$ ,

$$a_E(u, v) = \int_{\Omega} u_{i,j} c_{ijkl} v_{k,l} dx \quad (3)$$

For elasticity, each shape function is identified with a node and with one of the displacement fields, so the basis functions are

$$e_i g_a(x)$$

where  $e_i$  is the unit vector of displacement in coordinate direction  $i$ . So to compute the term of the stiffness matrix for displacement field  $i$  on shape function  $a$  and displacement field  $j$  on shape function  $b$ , substitute in (3)

$$u_i(x) = e_i g_a(x), \quad v_j(x) = e_j g_b(x)$$

Then (2) becomes

$$K_{iakb} = \int_E c_{ijkl} g_{a,j} g_{b,l} dx$$

which is then calculated as in (1.2)

**Remark 1** *This presentation is for the purpose of exposition only. In practical computation, the evaluation of the sums in the integrals is made more efficient by collapsing the index pairs  $i_j$  and  $kl$  into one index and using symmetries to reduce the number of calculations.*