Row Reduction

(Math 3191, Section 003, Spring 2000)

The purpose of this exercise is to give you a formal procedure for row-reducing a matrix, with a minimal amount of writing. After some practice, this procedure will become second-nature to you, so you will be able to do it without thinking. **NOTE: this procedure is slightly different (in step 2) than the one on pages 17-20 of the text.**

We begin with a matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \\ 8 & 16 & -44 & 24 & 63 & 138 \end{bmatrix}$$

Step 1: Locate the leftmost column that does not consist entirely of zeros. Call this the pivot column (in this case, it is the first column).

Step 2: Choose a row with a nonzero entry in the pivot column and interchange this row with the top row, so that a nonzero will be in the pivot position. Technically, it is O.K. to choose any row for the interchange; *however, for the sake of consistency, please choose the topmost row with a nonzero entry in the pivot position. (Thus, you will not do an interchange if you already have a nonzero pivot to start with)*:

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \quad \text{Originally row 2}$$

Step 3: Create zeros in the pivot column below the pivot by adding multiples of the pivot row to the remaining rows:

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \\ 0 & 0 & -4 & 0 & 15 & 26 \end{bmatrix} \quad \begin{array}{c} \text{Originally row 2} \\ \text{originally row 1} \\ \text{row 3 plus -1 times the pivot row} \\ \text{row 4 plus -4 times the pivot row} \end{array}$$

Step 4: Now cover the top row of the matrix, and repeat Steps 1-3 on the remaining part of the matrix. Repeat this process until there are no nonzeros below the pivot row.

$$\begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{-10} & \mathbf{6} & \mathbf{12} & \mathbf{28} \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{c} \text{cover (ignore) this row} \\ \text{pivot column= third column} \\ \text{row 3 plus 5/2 times row 2} \\ \text{row 4 plus -2 times row 2} \end{array}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{-10} & \mathbf{6} & \mathbf{12} & \mathbf{28} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} & \mathbf{0} & \mathbf{7} & \mathbf{12} \\ 0 & 0 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} \text{cover (ignore) this row} \\ \text{cover this row} \\ \text{pivot column = row 5} \\ \text{row 4 plus -2 times row 3} \end{array}$$

The matrix is now in *row-echelon form.* **Question: how would you define row-echelon form?** You could stop here, and use *back substitution* to solve the system. But we will continue on to put the matrix into *reduced row-echelon form* first.

Step 5: Starting with the rightmost pivot, scale the pivot row to make the pivot entry equal to 1. Then create zeros above the pivot entry by adding multiples of the pivot row.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 0 & 4 \\ 0 & 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \text{row 1 plus -24 times row 3} \\ \text{row 2 plus -14 times row 3} \\ \text{scaled (multiplied) by 2} \end{array}$$

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \text{row 1 plus -5 times row 2} \\ \text{scaled by -1/2} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{scaled by } 1/2$$

The matrix is now in *reduced row-echelon form.* From this matrix we can easily "read off" the solution to the linear system.

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 + 7, \quad x_2 \text{ and } x_4 \text{ are free} \\ x_3 &= 1 \\ x_5 &= 2 \end{aligned}$$