Math 3191
Applied Linear Algebra

Lecture 23: Orthogonal Projections, Gram-Schmidt

Stephen Billups

University of Colorado at Denver

Orthonormal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthonormal set** if

1. It is *orthogonal*.
2. Each vector has length 1.

If the orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ spans a vector space W, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is called an **orthonormal basis** for W.

Orthogonal Matrices

Recall that $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = 1$.

Suppose $U = [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3]$ where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Then

$$U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3] = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{u}_3 \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{u}_3 \\ \mathbf{u}_3^T\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} =$$

It can be shown that $UU^T = I$ also. So $U^{-1} = U^T$ (such a matrix is called an **orthogonal matrix**). (NOTE: U must be square to be orthogonal).

THEOREM 6 An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

THEOREM 7 Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbf{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

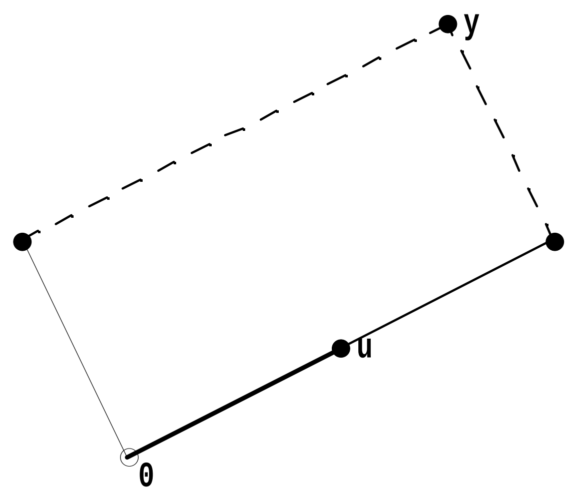
b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof of part b: $(U\mathbf{x}) \cdot (U\mathbf{y}) =$

Section 6.3 Orthogonal Sets

Review: $\widehat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is the **orthogonal projection of ____ onto _____**.

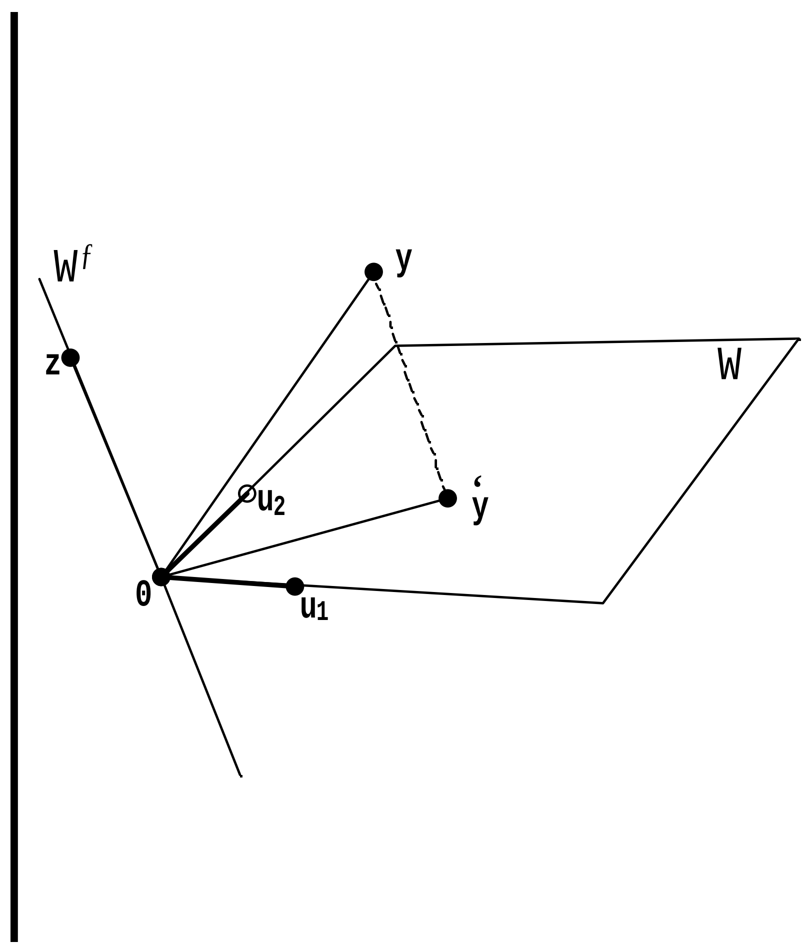


Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W in $\mathbf{R}^n$. For each $\mathbf{y}$ in W,

$$\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\right)\mathbf{u}_p$$

EXAMPLE

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbf{R}^3$ and let $W =$Span$\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ in $\mathbf{R}^3$ as the sum of a vector $\widehat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $W^{\perp}$.



Solution: Write

$$\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\right)\mathbf{u}_3$$

where

$$\widehat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2$$

$$\mathbf{z} = \left(\frac{\mathbf{y}\cdot\mathbf{u_3}}{\mathbf{u_3}\cdot\mathbf{u_3}}\right)\mathbf{u}_3.$$

To show that $\mathbf{z}$ is orthogonal to every vector in W, show that $\mathbf{z}$ is orthogonal to the vectors in $\{\mathbf{u}_1, \mathbf{u}_2\}$.

Since

$$\mathbf{z}\cdot\mathbf{u_1} = \qquad = \qquad = \mathbf{0}$$

$$\mathbf{z}\cdot\mathbf{u_2} = \qquad = \qquad = \mathbf{0}$$

THEOREM 8

THE ORTHOGONAL DECOMPOSITION THEOREM

Let W be a subspace of $\mathbf{R}^n$. Then each $\mathbf{y}$ in $\mathbf{R}^n$ can be uniquely represented in the form

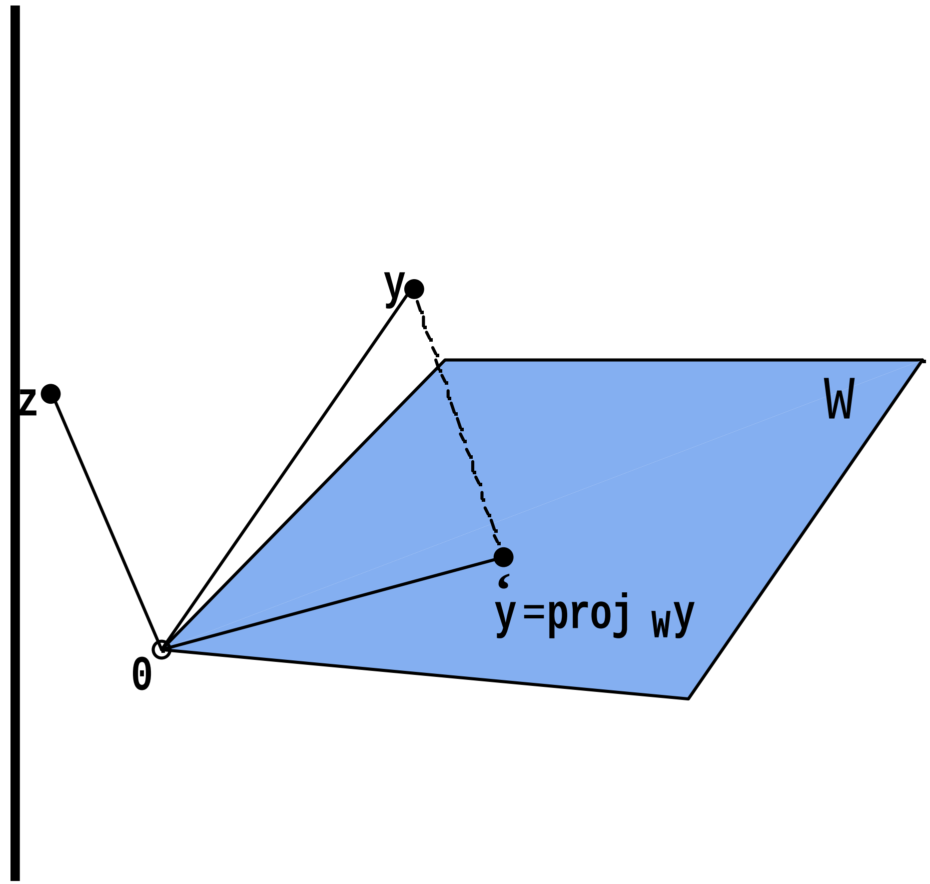
$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\widehat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\widehat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\right)\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}}$.

The vector $\widehat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto** W.


y
z
W
0
ŷ=proj Wy

EXAMPLE:

Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W =Span$\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

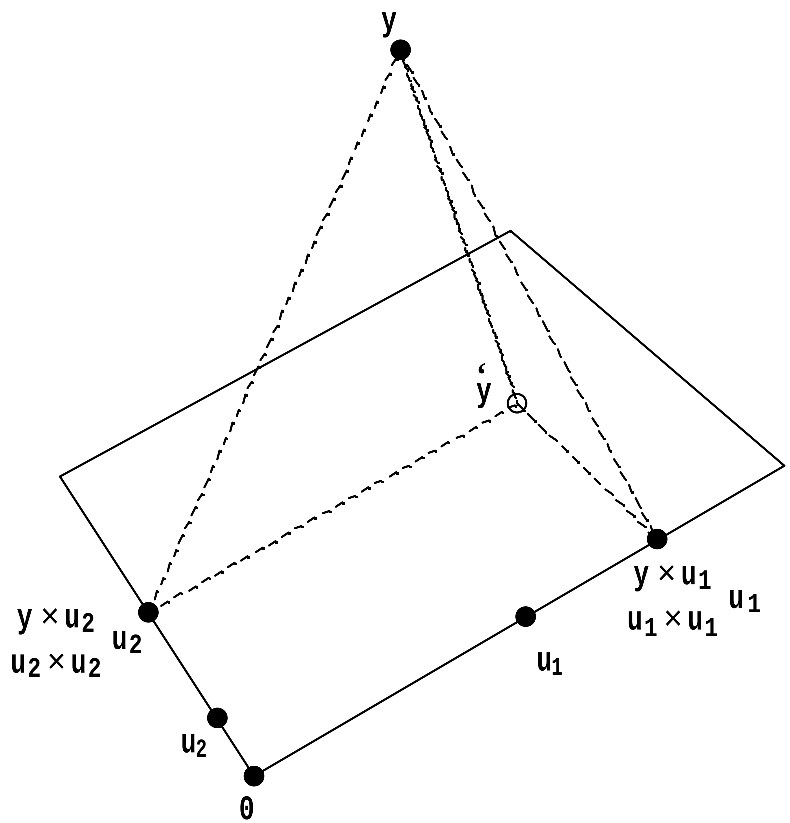
Solution:

$$\text{proj}_W \mathbf{y} = \widehat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2$$

$$= (\qquad) \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + (\qquad) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$$

Geometric Interpretation of Orthogonal Projections

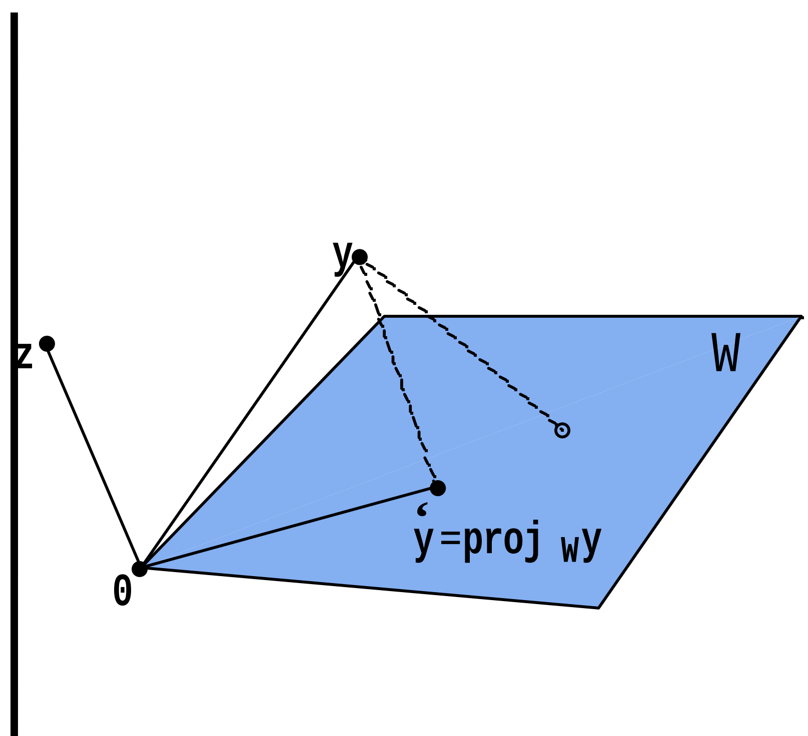


THEOREM 9 The Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, $\mathbf{y}$ any vector in $\mathbf{R}^n$, and $\widehat{\mathbf{y}}$ the orthogonal projection of $\mathbf{y}$ onto W. Then $\widehat{\mathbf{y}}$ is the point in W closest to $\mathbf{y}$, in the sense that

$$\|\mathbf{y}-\widehat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$$

for all $\mathbf{v}$ in W distinct from $\widehat{\mathbf{y}}$.



Outline of Proof

Let $\mathbf{v}$ in W distinct from $\widehat{\mathbf{y}}$. Then

$$\mathbf{v}-\widehat{\mathbf{y}} \text{ is also in } W \quad \text{(why?)}$$

$$\mathbf{z} = \mathbf{y}-\widehat{\mathbf{y}} \text{ is orthogonal to } W \quad \Rightarrow \quad \mathbf{y}-\widehat{\mathbf{y}} \text{ is orthogonal to } \mathbf{v}-\widehat{\mathbf{y}}$$

$$\mathbf{y} - \mathbf{v} = (\mathbf{y}-\widehat{\mathbf{y}}) + (\widehat{\mathbf{y}}-\mathbf{v}) \qquad \Longrightarrow \qquad \|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y}-\widehat{\mathbf{y}}\|^2 + \|\widehat{\mathbf{y}}-\mathbf{v}\|^2.$$

$$\|\mathbf{y} - \mathbf{v}\|^2 > \|\mathbf{y}-\widehat{\mathbf{y}}\|^2$$

Hence, $\|\mathbf{y}-\widehat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$. ■

EXAMPLE

Find the closest point to $\mathbf{y}$ in Span$\{\mathbf{u}_1, \mathbf{u}_2\}$ where

$$\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: $\widehat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2$

$$= (\qquad) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (\qquad) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} =$$

Another View of matrix Multiplication

Part of Theorem 10 below is based upon another way to view matrix multiplication where A is $m \times p$ and B is $p \times n$

$$AB = \begin{bmatrix} \text{col}_1 A & \text{col}_2 A & \cdots & \text{col}_p A \end{bmatrix} \begin{bmatrix} \text{row}_1 B \\ \text{row}_2 B \\ \vdots \\ \text{row}_p B \end{bmatrix}$$

$$= (\text{col}_1 A)(\text{row}_1 B) + \cdots + (\text{col}_p A)(\text{row}_p B)$$

For example

$$\begin{bmatrix} 5 & 6 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 34 & 5 & 3 \\ 10 & 3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & -2 \end{bmatrix}$$

$$=$$

So if $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$. Then $U^T = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T \end{bmatrix}$. So

$$UU^T = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_p\mathbf{u}_p^T$$

$$\left(UU^T\right)\mathbf{y} = \left(\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_p\mathbf{u}_p^T\right)\mathbf{y}$$

$$= \left(\mathbf{u}_1\mathbf{u}_1^T\right)\mathbf{y} + \left(\mathbf{u}_2\mathbf{u}_2^T\right)\mathbf{y} + \cdots + \left(\mathbf{u}_p\mathbf{u}_p^T\right)\mathbf{y}$$

$$= \mathbf{u}_1\left(\mathbf{u}_1^T\mathbf{y}\right) + \mathbf{u}_2\left(\mathbf{u}_2^T\mathbf{y}\right) + \cdots + \mathbf{u}_p\left(\mathbf{u}_p^T\mathbf{y}\right)$$

$$= (\mathbf{y}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\,\mathbf{u}_p$$

$$\Rightarrow \left(UU^T\right)\mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\,\mathbf{u}_p$$

THEOREM 10

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbf{R}^n$, then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\, \mathbf{u}_p$$

If $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$, then

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} \qquad \text{for all } \mathbf{y} \text{ in } \mathbf{R}^n.$$

Outline of Proof:

$$\text{proj}_W \mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\right)\mathbf{u}_p$$

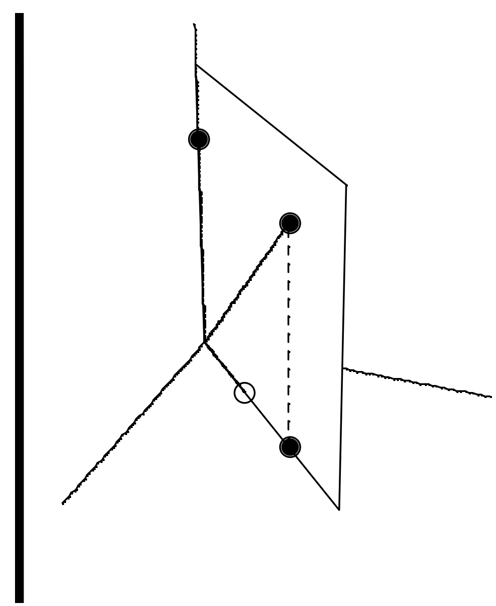
$$= (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\, \mathbf{u}_p = UU^T\mathbf{y}.$$

Section 6.4 The Gram-Schmidt Process

Goal: Form an orthogonal basis for a subspace W.

EXAMPLE: Suppose $W =$Span$\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.



Let

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\widehat{\mathbf{y}} = \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \widehat{\mathbf{y}} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

(component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$)

EXAMPLE

Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution: Let

$$\mathbf{v}_1 = \mathbf{x}_1 \text{ and } \mathbf{v}_2 = \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for Span$\{\mathbf{x}_1, \mathbf{x}_2\}$.

Let

$$\mathbf{v}_3 = \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \tfrac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

(component of $\mathbf{x}_3$ orthogonal to Span$\{\mathbf{x}_1, \mathbf{x}_2\}$)

Note that $\mathbf{v}_3$ is in W. Why?

$\Rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

Theorem 11: The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbf{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \tfrac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \tfrac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \tfrac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \tfrac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W and

$$\text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

EXAMPLE

Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, is a basis

for a

subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution: $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and

cont.

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix}$$

Replace $\mathbf{v}_2$ with $14\mathbf{v}_2 : \mathbf{v}_2 = 14 \begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$

(optional step - to make $\mathbf{v}_2$ easier to work with in the next step)

cont.

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{9}{630}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{70}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{bmatrix}$$

cont.

Rescale (optional): $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix}$

Orthogonal Basis for W:

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix} \right\}$$