Math 3191
Applied Linear Algebra

Lecture 12: Null and Column Spaces

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Announcements

- Study Guide 6 posted
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Subspaces, review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

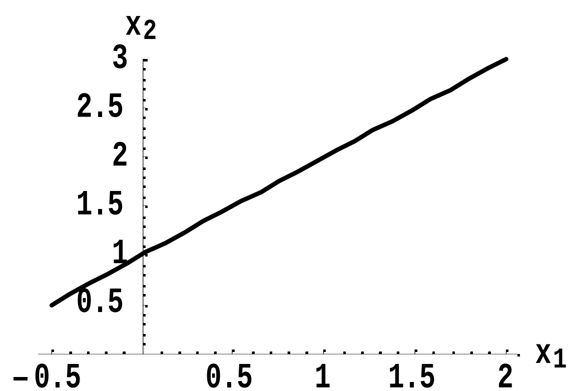
b. For each $\mathbf{u}$ and $\mathbf{v}$ are in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

Example

Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : \ x \text{ is real} \right\}$ a subspace of _______?

I.e., does H satisfy properties a, b and c?



Graphical Depiction of H

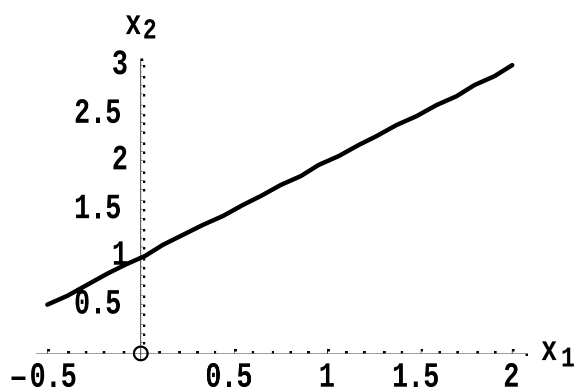
Solution

All three properties must hold in order for H to be a subspace of $\mathbf{R}^2$.

Property (a) is not true because

____.

Therefore H is not a subspace of $\mathbf{R}^2$.

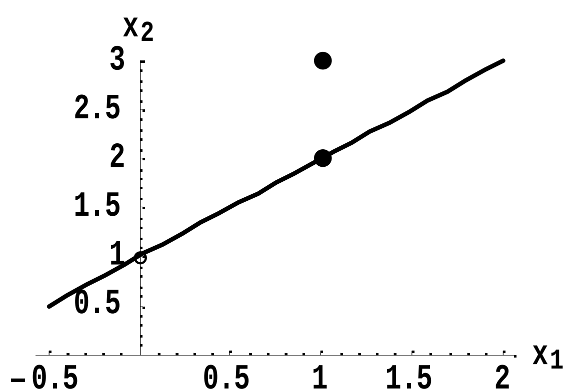


Another way to show that H is not a subspace of $\mathbf{R}^2$:

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is ____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



A Shortcut for Determining Subspaces

THEOREM 1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p.$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\underline{\quad\quad}\mathbf{v}_1 + \underline{\quad\quad}\mathbf{v}_1) + (\underline{\quad\quad}\mathbf{v}_2 + \underline{\quad\quad}\mathbf{v}_2) + \cdots + (\underline{\quad\quad}\mathbf{v}_p + \underline{\quad\quad}\mathbf{v}_p)$$

$$= (a_1 + b_1)\,\mathbf{v}_1 + (a_2 + b_2)\,\mathbf{v}_2 + \cdots + (a_p + b_p)\,\mathbf{v}_p.$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Proof (cont.)

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c\left(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p\right)$$

$$= \underline{\qquad\qquad}\mathbf{v}_1 + \underline{\qquad\qquad}\mathbf{v}_2 + \cdots + \underline{\qquad\qquad}\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Recap

1. To show that H is a subspace of a vector space, use Theorem 1. (I.e., rewrite H as the span of a set of vectors).
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

EXAMPLE:

Is $V = \{(a+2b, 2a-3b):\ a$ and b are real$\}$ a subspace of $\mathbf{R}^2$? Why or why not?

Solution: Write vectors in V in column form:

$$\begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix}$$

$$= \underline{\qquad\qquad} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underline{\qquad\qquad} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $V =$Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore V is a subspace of _____ by Theorem 1.

EXAMPLE:

Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : \ a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$? Why or why not?

Solution: $\mathbf{0}$ is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property ______ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

EXAMPLE:

Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a \begin{bmatrix} & \\ & \end{bmatrix} + b \begin{bmatrix} & \\ & \end{bmatrix}.$$

Therefore H $=$Span $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2\times 2}$.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the defi nition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since __________, $\mathbf{0}$ is in

__________.

Proof (cont).

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u} + \mathbf{v}$ is in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

$$\underline{\hspace{4cm}} \text{ and } \underline{\hspace{4cm}}.$$

Therefore

$$A(\mathbf{u} + \mathbf{v}) = \underline{\hspace{3cm}} + \underline{\hspace{3cm}} =$$
$$\underline{\hspace{3cm}} + \underline{\hspace{3cm}} = \underline{\hspace{1.5cm}}.$$

Property (c) If $\mathbf{u}$ is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \underline{\hspace{1cm}} A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an *explicit description of Nul A.*

EXAMPLE

Find an explicit description of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} == x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then Nul $A =$span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Question: What are $\mathbf{u}, \mathbf{v}, \mathbf{w}$?

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow c_1 = \_\_\_\_\_\_ \qquad c_2 = \_\_\_\_\_\_ \qquad c_3 = \_\_\_\_\_\_$$

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

Column Space

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \ \ldots \ \ \mathbf{a}_n]$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \ \ldots, \ \mathbf{a}_n\}}$$

THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Why? (Theorem 1, page 221)

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

EXAMPLE

Find a matrix A such that $W = Col\ A$ where $W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}$.

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore $A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$. (By Theorem 4, Chapter 1),

The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

Contrast between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col A. (There are infi nitely many possibilities.)

$$\_\_\_ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Contrast between Nul A and Col A (cont).

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x_1 = -2x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \end{array}$$

Let $x_2 =$ ___; then $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

Contrast Between Nul A and Col A where A is $m \times n$ (see page 232)

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

Review: THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

- If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.
- The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.
- The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

EXAMPLES:

(a) Determine whether the following set is a vector space or provide a counterexample.

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$$

Solution: *Since* ____ $= \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ is not in H, H is not a vector space.

Examples (cont)

(b) Determine whether the following set is a vector space or provide a counterexample.

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{c} x - y = 0 \\ y + z = 0 \end{array} \right\}$$

Solution: Rewrite $\begin{array}{c} x - y = 0 \\ y + z = 0 \end{array}$ as $\begin{bmatrix} & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

Examples

(c) Determine whether the following set is a vector space or provide a counterexample.

$$S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$$

One *Solution:* Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$; therefore S is a vector space by Theorem 1.

Examples

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S =$Col A where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space, since a column space is a vector space.

Kernel and Range of a Linear Transformation

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in tin V and all scalars c.

The *kernel* (or **null space**) of T is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. The *range* of T is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.

So if $T(\mathbf{x}) = A\mathbf{x}$, col $A =$ range of T.