MATH 2422 **FINAL EXAM–Solutions** Fall 2000

1. (25 pts) Let $\boldsymbol{u} = \langle 2, 2, 1\rangle$ and $\boldsymbol{v} = \langle 3, 0, -4\rangle$.

   (a) Calculate the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$. **Solution:** 2

   (b) Calculate the cross product $\boldsymbol{u} \times \boldsymbol{v}$. **Solution:** $\boldsymbol{\langle -8, 11, -6\rangle}$

   (c) What is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$? **Solution:**

$$\boldsymbol{\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|} = \frac{2}{15} \Longrightarrow \theta = \cos^{-1}(2/15).}$$

   (d) Find the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$. **Solution:**

$$\boldsymbol{\mathrm{proj}_v u = \frac{u \cdot v}{\|v\|^2} v = \frac{2}{25}\langle 3, 0, -4\rangle .}$$

   (e) Give a set of parametric equations for the line parallel to $\boldsymbol{v}$ passing through the point $(-1, 2, 5)$. **Solution:**

$$\boldsymbol{x = -1 + 3t, y = 2, z = 5 - 4t.}$$

2. (5 points) Write an equation of the plane that passes through the point $(-1, 1, 2)$ and is parallel to the plane determined by the equation $6x + y - 3z = 26$. **Solution:**

$$\boldsymbol{6(x+1) + (y-1) - 3(z-2) = 0} \quad \text{or} \quad \boldsymbol{6x + y - 3z + 11 = 0.}$$

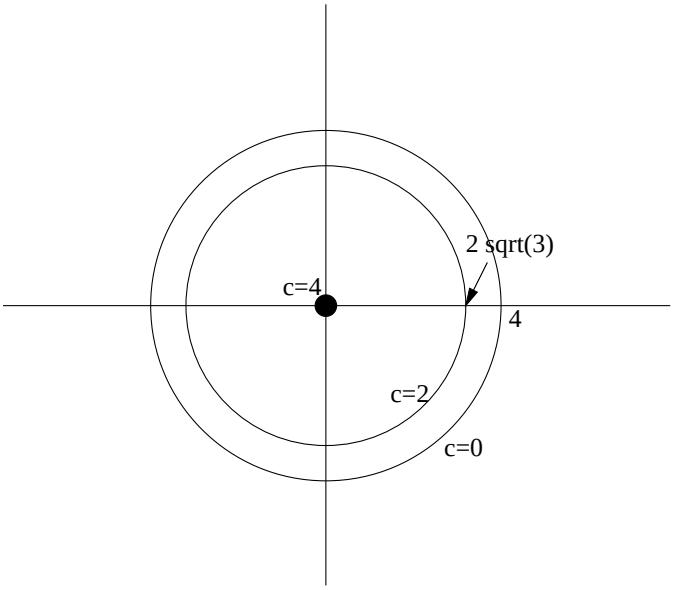
3. (5 pts) Let $x^2 + y^2 = 9$ be the equation (in rectangular coordinates) of a surface in three dimensions. Describe this surface as an equation using spherical coordinates. *Simplify the equation as much as you can.* **Solution:**

$$\boldsymbol{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 9 \Longrightarrow \rho^2\sin^2\phi = 9,} \quad \text{or} \quad \boldsymbol{\rho\sin\phi = 3.}$$

4. (5 pts) Find the velocity for an object having acceleration $\boldsymbol{a}(t) = e^t\boldsymbol{j} - 32\boldsymbol{k}$ and initial velocity $\boldsymbol{v}(0) = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$. **Solution:**

$$\begin{aligned} \boldsymbol{v(t)} &= \boldsymbol{e^t j - 32tk + C} \\ \boldsymbol{v(0)} &= \boldsymbol{j + C = 3i + 2j + k \Longrightarrow C = 3i + j + k} \\ \text{So } \boldsymbol{v(t)} &= \boldsymbol{3i + (e^t + 1)j + (1 - 32t)k.} \end{aligned}$$

5. (5 points) Sketch the level curves of the function $f(x,y) = \sqrt{16 - x^2 - y^2}$ for $c = 0, 2, 4$. *Label each curve with the corresponding value of c, and label where each curve crosses the x-axis.* **Solution:**



6. (20 points) Let $\boldsymbol{r}(t) = 2\sin(3t)\boldsymbol{i} + 2\cos(3t)\boldsymbol{j} + 8t\boldsymbol{k}$ represent the position of an object at time t.

   (a) What is the speed of the object at time $t = \pi/6$? **Solution:**

$$\begin{aligned} \boldsymbol{r}'(t) &= \mathbf{6\cos(3t)i - 6\sin(3t)j + 8k} \\ \|\boldsymbol{r}'(t)\| &= \mathbf{\sqrt{36\left(\cos^2(3t) + \sin^2(3t)\right) + 64} = \sqrt{36+64} = 10.} \end{aligned}$$

   (b) Calculate the arclength of $\boldsymbol{r}(t)$ over the interval $0 \le t \le \pi/6$. **Solution:**

$$\boldsymbol{s = \int_0^{\pi/6} \|r'(t)\|\, dt = \int_0^{\pi/6} 10 dt = 5\pi/3}$$

   (c) What is the curvature of $\boldsymbol{r}(t)$ at time $t = \pi/6$? **Solution:**

$$\begin{aligned} \boldsymbol{T}(t) &= \mathbf{\frac{r'(t)}{\|r'(t)\|} = \frac{3}{5}\cos(3t)i - \frac{3}{5}\sin(3t)i + \frac{4}{5}k.} \\ \boldsymbol{T}'(t) &= \mathbf{-\frac{9}{5}\sin(3t)i - \frac{9}{5}\cos(3t)j} \\ \boldsymbol{K} &= \mathbf{\frac{\|T'(t)\|}{\|r'(t)\|} = \frac{9}{50}.} \end{aligned}$$

   (d) Calculate the normal component of acceleration at time $t = \pi/6$. **Solution:**

$$\boldsymbol{a_N = K(\text{speed})^2 = \frac{9}{50}100 = 18.}$$

7. (25 points) Let $f(x,y,z) = xe^{yz}$

   (a) Calculate f_{xy}. **Solution:** $\boldsymbol{ze^{yz}}$

   (b) Find the total differential of f. **Solution:**

$$\boldsymbol{df = f_x dx + f_y dy + f_z dz = e^{yz} dx + xze^{yz} dy + xye^{yz} dz.}$$

   (c) Find the gradient of f at the point $(2,0,-4)$. **Solution:**

$$\boldsymbol{\nabla f = \langle f_x, f_y, f_z \rangle = \langle 1, -8, 0 \rangle}$$

   (d) Find the directional derivative of f at the point $(2,0,-4)$ in the direction of the vector $\boldsymbol{u = \frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k}$. **Solution:** $\boldsymbol{\|u\| = 1}$ `so` $\boldsymbol{u}$ `is a unit vector. Thus,` $\boldsymbol{D_u f = \nabla f \cdot u = 10/3.}$

   (e) Find the maximum value of the directional derivative at the point $(2,0,-4)$. **Solution:** $\boldsymbol{\|\nabla f\| = \sqrt{65}.}$

8. (5 points) Use the appropriate Chain Rule to find $\dfrac{\partial z}{\partial u}$ for $z = \sin x \cos y$, where $x = u - v$, and $y = u^2 + v^2$. **Write your answer as a function of u and v. Solution:**

$$\begin{aligned} \boldsymbol{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}} &= \boldsymbol{(\cos x \cos y)(1) - (\sin x \sin y)(2u)} \\ &= \boldsymbol{\cos(u-v)\cos(u^2+v^2) - \sin(u-v)\sin(u^2+v^2)(2u).} \end{aligned}$$

9. (5 points) At a certain instant, the height of a right circular cone is 30 inches and is increasing at a rate of 2 inches per second. At the same instant, the radius of the base is 20 inches and is increasing at the rate of 1 inch per second. At what rate is the volume increasing at that instant? (Recall $V = \frac{1}{3}\pi r^2 h$). **Solution:**

$$\boldsymbol{\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \left(\frac{2}{3}\pi r h\right)(1) + \left(\frac{1}{3}\pi r^2\right)(2) = \frac{2000\pi}{3}.}$$