Interdisciplinary Lively Application Project
Pendulum Activity

Title: Pendulum Activity

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Course Classification: Calculus 1 (MATH 1401 at CU-Denver),

Disciplinary Classifications: Mathematics, Physics

Prerequisite Skills:

1. Familiarity with Trigonometric Functions
2. Transformation of Functions
3. Differentiation and integration and relationship with distance, velocity, and acceleration
4. Graphing

Mathematics Classifications: Modeling, differentiation, integration, ODE

Physics Classifications: Speed, velocity, acceleration, gravity, Newton's second law of motion.

Materials Required:

1. TI-calculator with Calculator Based Ranger (CBR) used to measure distance (needed for activity I only). Instructor uses this only and then data should be provided to students.
2. String (approximately 5 ft for each group)
3. Tape (to tape string to ceiling)
4. ball
5. Stopwatch (for investigative segment)
6. Tape measure (for investigative segment)
7. Graphing or engineering paper (2 or 3 per person)
8. Scissors (for cutting string).

Interdisciplinary Lively Application Project:

Pendulum Activity

Introduction In the first half of this activity (Part A) we are going to explore the characteristics of a pendulum by using empirical (experimental) data. We will build a simple pendulum using a string and a bob (e.g. washer/ball), and determine which variables affect the period of the pendulum. Next you will use data to determine the horizontal displacement, and then use your experience to construct a function that represents the vertical displacement. We will then apply some calculus concepts to the horizontal displacement function to explore topics such as velocity and acceleration.

In the second half of the activity (Part B) we will apply Newton's second law of motion to obtain an analytical model (a model which is developed without experimental data) for the pendulum. This analytic model will be used to obtain an analytic expression for the period that confirms the results of Part A.

Although you will be performing the experiments as a group, each individual must turn in his/her own report. The report should include

1. A title page with names of group members (this could be the individual names of a small group, or if it's the entire class, put the name and section of the class).
2. A short introduction containing objectives of the activity.
3. A discussion which includes the task items. This should be written in proper English.
4. A conclusion.

The report may be handwritten or typed. It must be written with perfect spelling and grammar. It is *not* enough to turn in equations and graphs with no explanations (this sort of report will receive a failing grade). Graphs and equations should be motivated (why are we doing this?) and an explanation must be provided on how each new equation is derived.

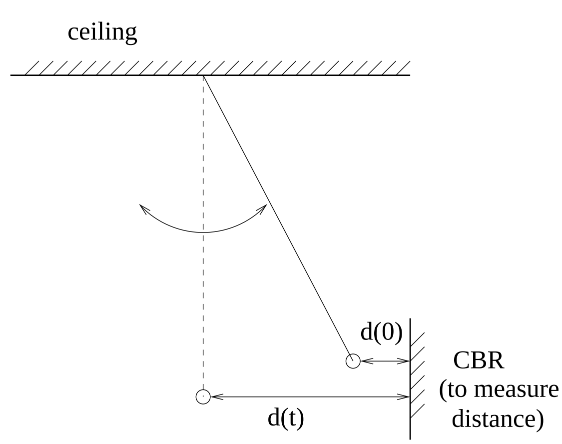

Figure 1: Pendulum Set-Up

Part A
I. Data Collection and Modeling of the Pendulum Motion

Horizontal Displacement of Pendulum

Your instructor will collect data on the horizontal displacement of the pendulum as it swings back and forth (see Figure 1) and transfer the distance versus time data to your calculator. (Note: when the CBR program collects the data, the distance is measured in meters and for this activity, the data needs to be converted to centimeters.) Be sure to remind your instructor that you need to know the length of string (or pendulum arm) so that you can check the relationship between the pendulum arm and period. The data appears to produce a sinusoidal pattern, so we will attempt to fit the data with a sinusoidal function of the form:

$$d(t) = A\cos(B(t - C)) + D. \quad (1)$$

Student Task #1

1. Begin by finding the period of the displacement function using two approaches. How many complete periods of a cosine function do you see? By tracing the function, record the time of the first period to 2 decimal places. Then trace all the complete periods and record the time to 2 decimal places. Use this value to calculate the time of one complete period to 2 decimal places. These two values obtained for the period should be very close. Record the value of the period you will use in the remaining calculations, and in your report explain how you chose this value.

2. Continue to trace on the data to calculate the Phase Shift, Amplitude, and Vertical Shift of the cosine function. Use these results to find the values for A, B, C and D in the displacement function. Formally write a summary of how you determined these values and then write the displacement function, $d(t)$, using your values for A, B, C, and D. To verify that you obtained the correct function, use your TI-89 calculator to graph and print the data as a scatter plot with your function superimposed on it. If it doesn't fit the data then check your work.

(continued)

Student Task #1 (continued)

3. Now suppose that we try to fit the data with a sine function instead of a cosine function:

$$d(t) = A\sin(B(t - C)) + D. \tag{2}$$

Explain how you obtain the values of A, B, C, and D for this function. Verify that this function produces the same curve as your cosine function.

4. According to Newton's second law of physics, the period of a pendulum can be found from the following formula which you will derive in Part B of this report:

$$\text{Period} = 2\pi\sqrt{\frac{L}{g}} \tag{3}$$

where $g = 9.81$ m/s^2 is the acceleration due to gravity and L is the length of the pendulum. What is g in units of cm/s^2? Using this formula what is the period of your pendulum? How does this value compare with your data?

Vertical Displacement of Pendulum

The vertical displacement function will be obtained from the horizontal pendulum data instead of being measured. The following worksheet is to be used as a guide. Be sure to save this information so that you can include it in your written report. You will not be turning in these data sheets.

First, consider the coordinate system positioned as in Figure 2 with the origin where the bob would be when the pendulum is at rest.

Let $y(t)$ be the distance (in cm) of the pendulum from the x-axis with respect to time, t, in seconds. As the bob swings back and forth, we assume the distance, $y(t)$, will be sinusoidal in shape.

1. From your horizontal displacement data, what is x_{max}? (Use Figure 2.) ______ cm.
2. Using x_{max} and L, calculate y_{max}. ______ cm.
3. What is y_{min}? ______ cm.
4. The period of the vertical displacement function is *not* the period of the pendulum (why?). How are the two periods related?

   ____.

5. Give two consecutive times, t_1 and t_2, when y reaches its maximum. $t_1 =$ ______sec. $t_2 =$ ______sec.

The period of the pendulum is ______s.

The period of $y(t)$ is ______s.

Now with your group sketch a graph (on graph/engineering paper) of the first 2 periods of the function, $y(t)$, which represents the distance from the x-axis as a function of time t. Corresponding to our displacement function we want $y(0)$ to be y_{max} when the pendulum is closest to the CBR. Remember, all graphs and equations should be included in your report. Assume no damping (i.e. assume the amplitude remains constant). Begin by locating and labeling the initial point $(0, y_{\text{max}})$ on your graph. Then locate the point when the bob again reaches its maximum height and label this point on your graph. Continue to plot and label the important points until you have shown two complete periods of the sinusoidal function.

Using a cosine function and information regarding vertical shift, amplitude, period, phase shift, and reflection, determine the function $y(t)$ (round decimals to 0.001).

$$y(t) = \underline{\hspace{6cm}} \tag{4}$$

Sketch $y(t)$ on your calculator/computer with a window similar to your graph (by hand) to check your equation. Be sure to print the plot for your report.

One way to check your function $y(t)$ is to graph $x(t)$ (which can be obtained from your function $d(t)$) and $y(t)$ in parametric mode. For a TI-89 calculator: Under MODE, be sure your Graph is in PARAMETRIC mode. Enter your functions in $y =$. While you are in this screen, go to Style (F6) and put it on Animate (5). Hit Return to change the style to Animate. Then Graph.

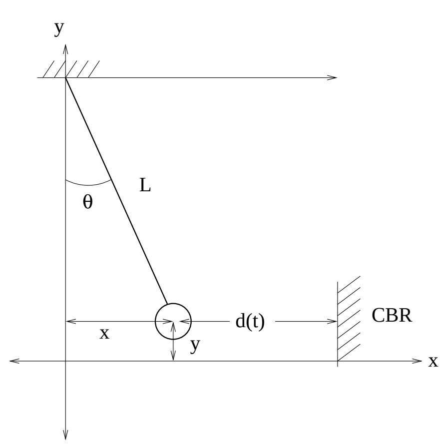

Figure 2: Coordinate System for Pendulum

II. Using the Model of the Pendulum Motion

Using your $d(t)$ of equation (1) from Part I: How far from the CBR is the pendulum **1 second** after it was released? ______. **5 seconds?** ______.

Determine the velocity and acceleration functions.

$$v(t) = d'(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$a(t) = v'(t) = d''(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(Note: You could check your answer with a computer algebra system).

Plot and print **two periods** of $v(t)$ and $a(t)$. Label the graphs and label the important points. Be sure to include these graphs in your report.

How fast is the pendulum moving 2 seconds after it was released? (include units) ______.
Is it moving **toward** or **away from** the CBR at this time? ______________________________
Explain. ____.

Let's use the graphs (the two periods) to answer some questions about the pendulum.

1. Over what time intervals is the pendulum moving toward the CBR?
   ____.

2. When is the pendulum moving the fastest? $t =$ ______. Explain:
   ____.

Include these answers in your report.

The **speed** of the pendulum can be represented by the **absolute value of the velocity function.** Plot and print $|v(t)|$. Be sure to include your graph in your report.

The pendulum is speeding up when the "speed function" is **increasing**. Over what time intervals is the pendulum speeding up?

____.

Now let's see what the velocity function and the acceleration functions are doing on these time intervals. Deactivate the "speed function" and activate the velocity and acceleration functions for plotting. Rewrite the intervals above and tell whether the velocity and acceleration functions are **positive** or **negative** at these times.

Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________

The pendulum is slowing down when the "speed function" is decreasing. Activate (plot) the "speed function" again. Over what time intervals is the pendulum slowing down?
____.

Deactivate the speed function and activate (plot) the velocity and acceleration functions one more time. Rewrite the intervals above and tell whether the velocity and acceleration functions are positive or negative at these times.

Interval __________ Velocity __________ Acceleration __________

Interval __________ Velocity __________ Acceleration __________

Interval __________ Velocity __________ Acceleration __________

Interval __________ Velocity __________ Acceleration __________

What are the patterns that you see. Include these patterns/conclusions in your report.

____.

____.

____.

____.

____.

Be sure to save your plots and your collected data for use in the report that you must write.

Student Task #2

Formally write up a summary of what was accomplished in this section. Include your graphs and explanations. Write it up so that a reader who has not performed the experiment nor seen these activity sheets could follow what was done and why. Do *not* turn in these papers, these are to be used to help organize your report.

Part B

The motion of many objects (the orbit of Earth about the sun, the trajectory of a baseball, the deformation of a spring) can be described using Newton's second law of motion. This law states that *Force equals mass times acceleration*. Our goal is to derive the equation for $d(t)$ and $y(t)$ from Newton's second law using reasonable assumptions We then compare the results of this model with the results obtained in Part A, which were based purely on experimental data.

I. Analytical Model of the Pendulum Motion

Shown in Figure 3 is a pendulum consisting of a bob with mass m attached to a pivot by a massless string of length L. The important observation is that the string constrains the motion of the pendulum bob so that there is no motion in the *radial* direction along the string. All of the motion is in the *tangential* direction, which is always perpendicular to the string. (Equivalently, the tension in the string exactly balances the weight of the bob in the direction of the string.)

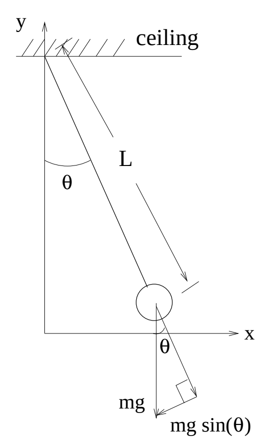

Figure 3: The simple pendulum consists of a bob with mass m attached to a pivot by a string of length L. The component of the bob's weight in the θ direction is $-mg\sin\theta$.

The fact that the motion is entirely in the tangential direction means that we can describe the motion of the pendulum using the variable $\theta(t)$ that gives the angle between the string and the vertical (see Figure 3). We will agree that positive values of θ correspond to displacements to the right of the vertical line.

We must now write Newton's second law, *Force equals mass times acceleration*, in the θ direction. The only fact that we must take on faith is that the acceleration of the bob in the θ direction is $L\frac{d^2\theta}{dt^2}$. Notice that the second derivative term, $\frac{d^2\theta}{dt^2}$, looks like an acceleration. We must multiply the second derivative by the length of the pendulum, L, so that the units are Length/Time^2, which are correct for an acceleration.

Let's now consider the external forces acting on the pendulum bob: Because we neglect air resistance and friction, the only force acting on the bob is gravity, which is reflected in the weight of the bob, mg.

Using the right triangle shown in Figure 3, show that the component of the weight in the tangential direction is $-mg\sin\theta$.

Notice that the componenet of the weight acting in the tangential direction is always negative. A negative sign appears on this force because it always acts in the direction toward the y-axis - it acts in the negative θ direction when $\theta > 0$ ($\theta > 0$ implies that $\sin(\theta) > 0$ implies that $-mg\sin(\theta) < 0$) and in the positive θ direction when $\theta < 0$ ($\theta < 0$ implies that $\sin(\theta) < 0$ implies that $-mg\sin(\theta) > 0$).

With these observations, Newton's second law becomes

$$\text{mass} \times \text{acceleration} = \text{force in tangential direction}$$

or

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta.$$

Of great significance, the mass cancels from both sides, which means that in the absence of air resistance and friction, the motion is independent of the mass of the bob. Rearranging terms a bit, the equation of motion may be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

This equation is a *nonlinear differential equation* and is difficult to solve. One way to make the equation easier to solve is to approximate $\sin\theta$ by θ. To see why this approximation can be made, graph $y = \sin\theta$ and $y = \theta$. For what values of x are the graphs close to one another?

Using this approximation ($\sin\theta \approx \theta$ for small values of θ) the above nonlinear equation can now be written as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{5}$$

You will learn a rigorous method for finding a solution to this sort of differential equation in class, but for now we will give you the solution and it will be your task to show that this is the solution. First, there are lots of solutions to (5). Recall that for a first-order differential equation (e.g. $y' = 5$) we get one arbitrary constant, (e.g. $y = 5x + C_1$), and for a second-order differential equation (e.g. $y'' = 5$) we get two arbitrary constants (e.g. $y = 5/2x^2 + C_1x + C_2$). So to get exactly one solution to (5) we need to provide two *initial conditions*, or the state of the pendulum at time $t = 0$. Suppose we start the pendulum from rest (i.e. not moving: $\frac{d\theta}{dt}(0) = 0$) at a given initial angle ($\theta(0) = \theta_0$). Then formally we say that we must find the solution to the following differential equation subject to the initial conditions:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \tag{6}$$

$$\theta(0) = \theta_0 \qquad \frac{d\theta}{dt}(0) = 0. \tag{7}$$

Student Task #3a Derive equation (6) as was done in this section using your own words.

Student Task #3b
Show that the function

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right) \tag{8}$$

is a solution to differential equation (6) and that it also satisfies the initial conditions in equation (7).

Student Task #4
Show that the period of the motion given in student task #3 above is

$$\text{Period} = 2\pi\sqrt{\frac{L}{g}}.$$

(Note: This only takes one or two steps!)

One way of obtaining a solution to the differential equation (6) with initial conditions (7) is to 'guess' the form of the solution, and then determine the coefficients in the guessed form:

Student Task #5
'Guess' that the form of the solution to the differential equation (6) with initial conditions (7) is:

$$\theta(t) = A\cos(\alpha t) + B\sin(\beta t). \tag{9}$$

By applying this equation to the initial conditions (equation (7)) and the differential equation (equation (6)) show that the result is the solution given in Student Task #3. That is, show that

1. A in equation (9) must equal θ_0 (use equation (7)),
2. B in equation (9) must equal 0 (use equation (7)), and
3. α in equation (9) must equal $\sqrt{\frac{g}{L}}$ (use equation (6)).

Finally, in order to compare these results with the experiments of Part A, we must recover the x and y coordinates of the pendulum bob.

Student Task #6

1. Use Figure 3 and right triangles to show that

$$x(t) = L\sin(\theta(t)) \qquad y(t) = L - L\cos(\theta(t)) \tag{10}$$

   where L is the length of the pendulum string, which we assume to be a constant.

2. Write the expression for $d(t)$ and $y(t)$ in the two pendulum experiments of Part A using the values of L from your experiment. Are the results similar? Explain.

Open Discussion: In engineering problems, engineers must design things (engines, buildings, bridges, computer chips) at a minimal cost. Sometimes it is too expensive to perform real-time experiments on an object (will a particular building withstand an earthquake of magnitude 6?), so that the engineer must make design specifications based purely on a physics-based model with reasonable assumptions. Safety factors are then built in to account for these assumptions. Other problems may allow engineers to perform experiments which complement a physics-based model (e.g. using a wind tunnel to help design an airplane wing), and other problems (e.g. when dealing with soil which may vary markedly over short ranges) may require a model to be developed purely on experiments since trying to develop a physics-based model becomes too complicated to solve, even with today's computational power.